Allocating heterogeneous goods through wait times and prices

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Motivation: affordable housing

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- Applicants have **heterogeneous preferences** over location and quality
- More desirable units come with **longer wait times** and **higher rents**
- When choosing which project to apply for, applicants trade off:
  1. Preferences for different locations/apartments
  2. Wait times
  3. Rents
- While implementations differ, this **core trade-off** present across programs

Question

- **Question:** How should we screen with wait times and prices when allocating heterogeneous goods?

- I study a stylized model with two goods and two screening instruments

- **Main result:** The designer should only use pricing to screen, even if she has no value for revenue

Literature

- **Wasteful screening** (Hartline and Roughgarden, 2008; Yang, 2021)
  - **This paper**: combines two wasteful screening instruments
- **Wait times 'acting as prices'** (Barzel, 1974; Leshno, 2022; Ashlagi et al., 2024)
  - **This paper**: shows wait times and prices screen on different things
- **Mechanisms without money** (Hylland and Zeckhauser, 1979; Budish, 2011)
  - **This paper**: money allowed but transfers are wasteful

Model

Goods

- The designer distributes two kinds of goods, $\boldsymbol{A}$ and $\boldsymbol{B}$,
  - There is $\boldsymbol{\mu_A} > 0$ of good A and $\boldsymbol{\mu_B} > 0$ of good B
- Agents' values for A and B given by two-dimensional types $\boldsymbol{(a, b)} \in [0,1]^2$
  - Values a and b distributed independently on $[0,1]$, according to $\boldsymbol{G}$ and $\boldsymbol{H}$
  - G, H have densities g, h, full-support, $\frac{G(v)}{g(v)}, \frac{H(v)}{h(v)}$ strictly increasing

Agents

- The designer chooses a menu of wait times and payments for each of the goods
  - Each agent chooses which good she wants (if any) …
  - … and then chooses a payment and wait time option from this good's menu
- Type-(a, b) who gets a good, pays $\boldsymbol{p}$ and discounts it by $\boldsymbol{x}$ due to waiting gets utility:

$$\begin{aligned} &x \cdot a - p \quad \text{if she gets } A, \\ &x \cdot b - p \quad \text{if she gets } B. \end{aligned}$$

- NB: waiting **delays receipt** $\Rightarrow$ waiting cost **multiplies value** for the good!

Allocations

- The designer chooses **allocations** of:

  1. Payments $p : [0,1]^2 \to \mathbb{R}_+$
  2. Discounting $x : [0,1]^2 \to [0,1]$
  3. Goods: $y : [0,1]^2 \to \{A, B, \varnothing\}$

- Subject to **IC, IR** and **supply** constraints:

$$\text{for every } (a,b),(a',b') \in [0,1]^2, \quad U[a,b,(p,x,y)(a,b)] \geq U[a,b,(p,x,y)(a',b')] \tag{IC}$$

$$\text{for every } (a,b) \in [0,1]^2, \quad U[a,b,(p,x,y)(a,b)] \geq 0 \tag{IR}$$

$$\int \mathbb{1}_{\text{gets } A}\, \mathrm{d}F(a,b) \;\leq\; \mu_A, \qquad \int \mathbb{1}_{\text{gets } B}\, \mathrm{d}F(a,b) \;\leq\; \mu_B \tag{S}$$

Designer

- She maximizes total agent welfare:

$$W = \int U[a, b, \ (p, x, y)(a, b)] \, \mathrm{d}F(a, b)$$

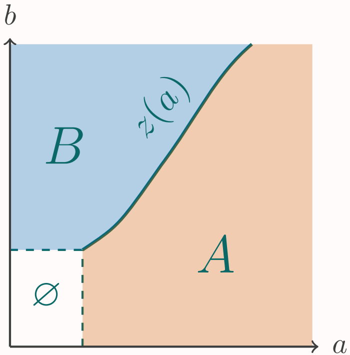
- NB: the designer puts **no value on revenue**!
  - E.g. social programs whose participants are poorer than the average taxpayer
  - Extreme assumption, but it works *against* the main result!

- Technical restriction: allowing only piecewise diff-able discounting allocations $x(a, b)$

Feasible mechanisms

Who gets which good?

- When neither good is free, some types do not participate ($\varnothing$)
- The rest pick their favourite (payment, wait time) option for one of the goods
- Types on the **boundary $\boldsymbol{z}$** indifferent between their best options for both goods
- Types below z pick some option for A, types above z pick some option for B



Main result

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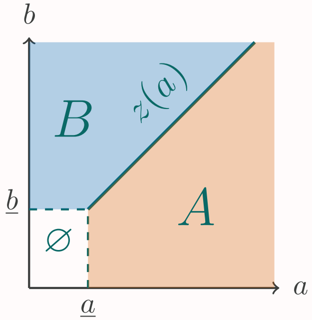
Theorem 1

The optimal mechanism allocates both goods **without waiting**. It posts a separate **price for each good**. The prices are chosen so that the whole **supply of both goods** is allocated.



Main result

Theorem 1

The optimal mechanism allocates both goods **without waiting**. It posts a separate **price for each good**. The prices are chosen so that the whole **supply of both goods** is allocated.

- I will give two complementary intuitions:
  - **Intuition 1**: explains why the result holds in a 1-dimensional case
  - **Intuition 2**: looks at what multidimensionality adds to the problem

Intuition 1: 1D case

1D case

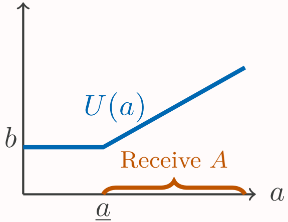
- Unit mass of agents with **same** $\boldsymbol{b} > 0$ (sufficiently small) and $\boldsymbol{a} \sim \boldsymbol{G}$ on $[0, 1]$
- Unlimited supply of good B, supply $\boldsymbol{\mu_A}$ of good A

Proposition 1

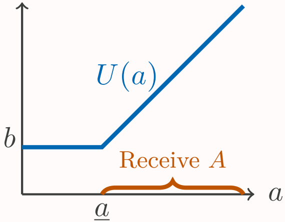
The optimal mechanism in the 1D model offers both goods **without waiting**. It offers **good $\boldsymbol{B}$ for free** and posts a **price for good $\boldsymbol{A}$**.

1D case

- Every feasible (deterministic) 1D mechanism allocates A to types above some $\underline{\boldsymbol{a}}$
- We can enforce this cutoff by asking recipients of A to **pay** or to **wait**
- We have $U(a) = b + \int_{\underline{a}}^{a} x(v)dv \ldots$



Wait times



Payments

- . . . so payments **leave more rents to inframarginal** takers of A!

1D case

- Wait time and payments mechanisms equally good for the cutoff type…
- …but wait times more costly to inframarginal types…
- …while payments 'equally costly' to everyone

- However, in **1D**, the A-good always goes to **an upper interval of types**
- In **2D**, combining wait times and payments can **change sorting into goods**!
- **Intuition 2** explains why payments sort agents better

Intuition 2: 2D case

Only wait times vs. only payments

Only wait times

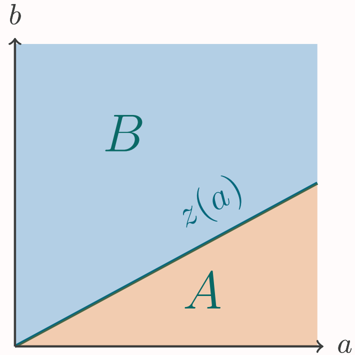
- Suppose $\mu_A + \mu_B = 1$ and both goods are given for free

- Then everyone joins and wait-times 'clear the market'

- Type (a, b) chooses A if:

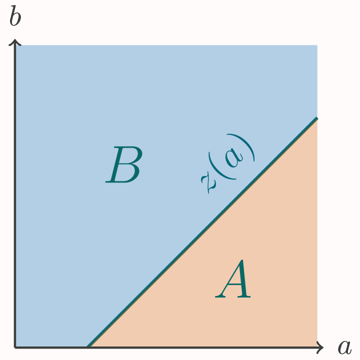
$$x_A \cdot a > x_B \cdot b$$

- **Ratio** $\frac{a}{b}$ determines choice of good



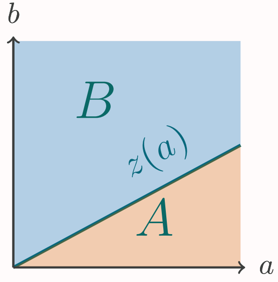
Only payments

- Suppose $\mu_A + \mu_B = 1$
- We can achieve the efficient allocation by pricing the overdemanded good!
- A-goods go to those with **highest** $\boldsymbol{a - b}$

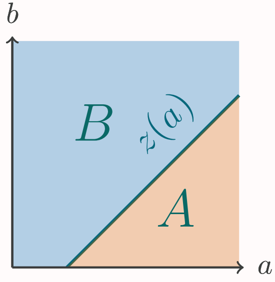


Using wait times vs. only payments

- With wait times, agents sort based on **relative values**
- Payments let us screen on agents' **absolute values**



Only wait times



Only payments

- **Absolute values** are what matters, so payments sort agents better!

Proof intuition

Indirect utilities

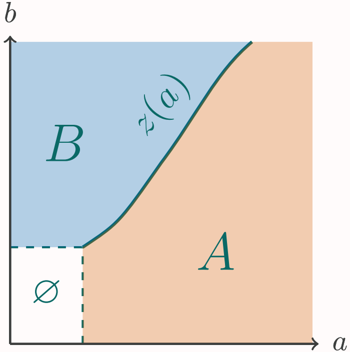
- Agents in the A-region choose some (wait time, payment) options for good A...

- ...and agents in the B-region choose one for good B

- Their option choice **does not depend** on their value for the other good!

- Therefore, we can write indirect utilities cond. on getting goods A and B as:

$$U_A(a),\ U_B(b)$$



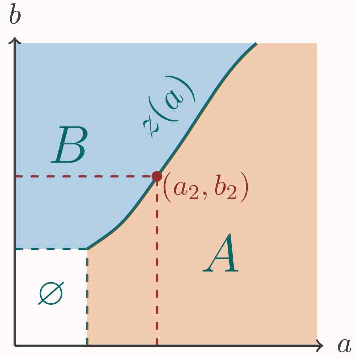
Indirect utilities

- $U_A(a), U_B(b)$ are the indirect utilities cond. on joining goods A and B

- We thus have two 1D screening problems (one for each good)...

- ...connected by the boundary types' indifference conditions:

$$U_A(a_2) = U_B(z(a_2))$$



Proof strategy

1. Fix any boundary z and find the mechanism that optimally implements it
2. Find the optimal z among optimally implemented boundaries

Optimally implementing a given boundary

- Fix a boundary z and recall the following holds along it:

$$U_A(a) = U_B(z(a)) \quad \Rightarrow \quad x_A(a) = x_B(z(a)) \cdot z'(a)$$

- We have $U_A(a) = \int_{\underline{a}}^{a} x_A(v)dv$, so we want x_A **as large as possible**

- Finding the best mechanism implementing $z \Leftrightarrow$ finding the p.w. largest **non-decreasing** $x_A, x_B : [0,1] \to [0,1]$ satisfying:

$$x_A(a) = x_B(z(a)) \cdot z'(a)$$

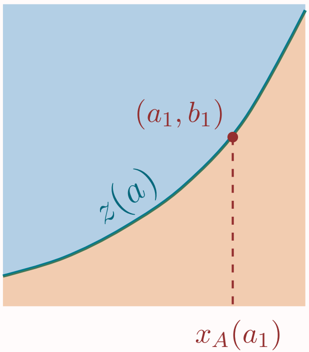
Picking the optimal boundary

- Fix some $x_A(a_1)$ and suppose z is convex below it. Then:

$$z'(a) = \frac{x_A(a)}{x_B(z(a))} \quad \nearrow$$

- So $x_A(a)$ must be strictly below $x_A(a_1)$ for $a < a_1 \ldots$

- Best we can do is to push both x_A and x_B up until **monotonicity binds** for x_B



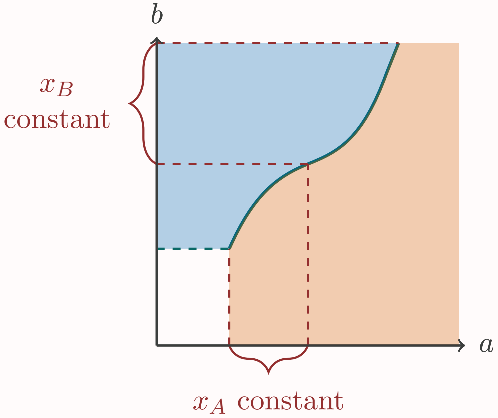
How flat can we make U_A and U_B?

- Thus, in the optimal mechanism...
- on **convex** regions we have:

$$x_B(z(a)) = \text{const}, \quad x_A(a) \propto z'(a).$$

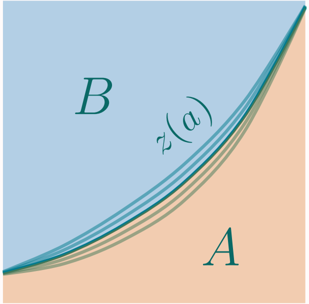
- and on **concave** regions we have:

$$x_A(a) = \text{const}, \quad x_B(z(a)) \propto 1/z'(a).$$



Picking the optimal boundary

- These conditions tell us **how to optimally implement** each boundary z
- Now, look at any convex region of z
- Perturb z to find its **optimal shape** on it



Objective in terms of U_A and U_B, and z

- Recall the objective is:

$$\int U\left[a, b, (p, t, y)(a, b)\right] \mathrm{d}F(a, b).$$

- We can use the boundary structure to write it as:

$$\underbrace{\int_{\underline{a}}^{1} \int_{0}^{z(\min[a,\overline{a}])} f(a, v) dv \cdot U_A(a)\ da}_{\text{Get } A} + \underbrace{\int_{\underline{b}}^{1} \int_{0}^{z^{-1}(\min[b,\overline{b}])} f(v, b) dv \cdot U_B(b)\ db.}_{\text{Get } B}$$

- We can similarly rewrite supply constraints in terms of z

Objective in terms of U_A and U_B, and z

$$\int_{\underline{a}}^{1} \int_{0}^{z(\min[a,\overline{a}])} f(a,v)dv \cdot U_A(a)\, da \;+\; \int_{\underline{b}}^{1} \int_{0}^{z^{-1}(\min[b,\overline{b}])} f(v,b)dv \cdot U_B(b)\, db.$$

- Restricting to some region $[\underline{v}, \overline{v}]$, changing variables and integrating by parts gives:

$$U_A(\overline{v})G(\overline{v})H(z(\overline{v})) - U_A(\underline{v})G(\underline{v})H(z(\underline{v})) - \int_{\underline{v}}^{\overline{v}} \underbrace{x_A(a)}_{\text{constant}} G(a)H(z(a))\, da.$$

- Objective depends only on z, so we can apply optimal control
- Turns out the optimal z has to be **linear on every such region**!

Conclusions

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- Many housing programs make participants trade off:

  prefs. for goods vs. **wait time** vs. **payments**

- These screen differently! Wait times → relative, payments → absolute values
- My stylized model shows wait-times have **bad screening properties**

- While some wait time is often inevitable in reality…
- …we should be worried about large **imbalances** in wait times!
- In those cases, we should **adjust prices**!

Thank you!