Ironing Without Concavification*

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Abstract

I propose a new approach to solving standard screening problems when the monotonicity constraint binds. A simple geometric argument shows that when virtual values are quasi-concave, the optimal allocation can be found by appropriately truncating the solution to the relaxed problem. I provide an algorithm for finding this optimal truncation when virtual values are concave.

1 Introduction

This note revisits the problem of finding optimal menus in standard single-agent screening environments. When agents' payoffs are quasi-linear and satisfy single-crossing, the standard approach involves writing the objective as an integral of "virtual values" which depend only on one type's allocation and separately choosing each type's allocation to maximize its corresponding virtual value. Incentive compatibility, however, requires that the allocation be non-decreasing in type. If this constraint binds, it is standard to transform virtual values so that after the transformation, pointwise maximization yields an increasing solution. This transformation, known as ironing, was described by Myerson (1981) and subsequently generalized by Toikka (2011). A different approach uses optimal control methods (Guesnerie and Laffont, 1984; Hellwig, 2008; Ruiz del Portal, 2011).

I propose an alternative approach that involves solving the relaxed problem without the monotonicity constraint and transforming the resulting allocation to satisfy monotonicity. Theorem 1 says that whenever an optimal allocation rule exists, it can be found by optimally truncating the solution to the relaxed problem. Moreover, the optimal truncation is pinned down by the allocations of types at which the solution to the relaxed problem changes monotonicity. These observations generalize insights about the structure of solutions from the literature—they require no continuity or differentiability assumptions, do not need virtual values to be concave, and do not assume that an agent's allocation is chosen from a compact interval. Therefore, in contrast to previous work, my results also apply when the planner can only assign discrete allocations.

I subsequently use these insights to develop a simple algorithm for finding the optimal allocation rule under the additional assumptions that virtual values are concave and that each agent's allocation is chosen from a compact interval.

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2 Problem

Agents with types $\theta \in [0,1]$ are assigned allocations from a compact set $\mathcal{X} \subset \mathbb{R}$. The planner chooses a non-decreasing allocation rule $x : [0,1] \to \mathcal{X}$ to maximize:

$$V[x] = \int_0^1 J(x(\theta), \theta) d\theta. \tag{1}$$

I refer to $J : \mathcal{X} \times [0,1] \to \mathbb{R}$ as the virtual value.

Assumption 1. *The virtual value satisfies the following properties:*

1. *$J(\cdot, \theta)$ is weakly quasi-concave for every $\theta \in [0,1]$.*
2. *$J(x, \theta)$ is uniformly bounded on $\mathcal{X} \times [0,1]$.*

I will call any x that maximizes $J(\cdot, \theta)$ pointwise a solution to the relaxed problem. I also assume that a well-behaved solution exists:

Assumption 2. *There exists a solution to the relaxed problem, x_R, that is piecewise monotonic.*

3 Example

I now present a simple screening problem in which virtual values are quasi-concave but not necessarily concave. Thus, the usual ironing approach does not apply to it in general.

A monopolist sells a service of quality $x \in [0, \bar{x}]$ to a single buyer. The buyer's type is $\theta \in [0,1]$, distributed according to F with a continuous density $f > 0$. A type-θ buyer who receives quality x and pays p gets utility

$$u(\theta, x, p) = x + \theta h(x) - p.$$

The seller's cost of providing quality x is $c(x)$. Both h and c are twice continuously differentiable with $h'(x) > 0$ for $x > 0$ and $h(0) = 0$. The seller chooses an incentive-compatible and individually rational mechanism to maximize expected profit.

Since the utility function satisfies single crossing, incentive compatibility requires $x(\theta)$ to be non-decreasing in θ. By the envelope theorem and integration by parts, the seller's relaxed objective can be written as (1) with

$$J(x, \theta) = f(\theta) \left[x + \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) h(x) - c(x) \right].$$

Fact 1. *Assume that $h'(0) = c'(0) = 0$, $h''(x) > 0$, $c''(x) > 0$ for $x > 0$, and that $\frac{c''(x)}{h''(x)}$ is weakly increasing in x. Then $J(\cdot, \theta)$ is quasi-concave for every θ.*

Proof. Fix θ and let

$$K := \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

Since $f(\theta) > 0$, it suffices to show that $x + Kh(x) - c(x)$ is quasi-concave in x. Its first and second derivatives are

$$1 + Kh'(x) - c'(x) \quad \text{and} \quad h''(x)\left[K - \frac{c''(x)}{h''(x)}\right].$$

Since $c''(x)/h''(x)$ is weakly increasing and $h''(x) > 0$, the set on which the second derivative is nonnegative is an initial interval. Hence the first derivative first weakly increases and then weakly decreases. Since its value at $x = 0$ is 1, it crosses zero at most once. Thus $x + Kh(x) - c(x)$, and hence $J(\cdot, \theta)$, is single-peaked and therefore quasi-concave. □

Moreover, for many functional forms, the resulting virtual values will not be concave in x for all θ. This is the case, for instance, for

$$h(x) = \frac{x^2}{2}, \qquad c(x) = \frac{x^3}{3}.$$

4 Structure of the solution

In this section I present Theorem 1 describing the structure of the solution to the planner's problem. I first introduce the following definition:

Definition 1. *A point $i \in (0,1)$ is a* ***critical point*** *if x_R changes monotonicity there, that is, if x_R is monotonic on $(i - \epsilon, i)$ for some $\epsilon > 0$ but is not monotonic on $(i - \epsilon, i + \delta)$ for any $\delta > 0$. By convention, I also call 0 and 1 critical points. I use i_n to denote the n-th critical point and $\mathcal{I}$ to denote the set of critical points.*

I also define the function $x_v^* : [0,1] \to \mathcal{X}$ parametrized by $v = (v_1, \ldots, v_{|\mathcal{I}|-1}) \in \mathcal{X}^{|\mathcal{I}|-1}$:

$$x_v^*(\theta) = \begin{cases} \min\{v_{n+1}, \max\{v_n, x_R(\theta)\}\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \end{cases} \tag{2}$$

where, by convention, $v_{|\mathcal{I}|} = x_v^*(1) = \max_{x \in \mathcal{X}} x$.

Theorem 1. *If the planner's problem has a solution, it has a solution of the form x_v^* for some $v \in \mathcal{X}^{|\mathcal{I}|-1}$. This solution can be recovered by solving:*

$$\max_{v \in \mathcal{X}^{|\mathcal{I}|-1}} V[x_v^*] \quad \textit{subject to} \quad v_1 \leq v_2 \leq \cdots \leq v_{|\mathcal{I}|-1}. \tag{I}$$

Theorem 1 says that when the planner's problem has a solution, we can find it by optimally truncating the solution to the relaxed problem, x_R. Moreover, the optimal truncation is pinned down by the allocations of types at which x_R changes monotonicity.

Proof of Theorem 1. I first prove three lemmas:

Lemma 1. *If x_1, x_2 are allocation rules and x_2 lies pointwise between x_1 and x_R, then $V[x_2] \geq V[x_1]$.*

Proof. $J(x_R(\theta),\theta) \geq J(x_1(\theta),\theta)$ for all θ by definition. Now, by quasi-concavity of $J(\cdot,\theta)$:

$$V[x_2] = \int_0^1 J(x_2(\theta),\theta)d\theta \geq \int_0^1 \min\{J(x_1(\theta),\theta), J(x_R(\theta),\theta)\}d\theta$$
$$= \int_0^1 J(x_1(\theta),\theta)d\theta = V[x_1].$$

□

Lemma 1 tells us that the objective always increases when we move the allocation rule pointwise closer to the first-best one. This property underpins the proofs of Lemmas 2 and 3, illustrated in Figures 1a and 1b.

Lemma 2. *Suppose x_R is decreasing on $[a,b)$. Then any increasing allocation rule x can be weakly improved upon by some allocation rule x^* that is constant on $[a,b)$ and coincides with x elsewhere.*

Proof. Fix any increasing x. Consider x^* that coincides with x on $[0,1] \setminus [a,b)$ and takes the following values for $\theta \in [a,b)$:

$$x^*(\theta) = \begin{cases} x(a^+) & \text{if } x_R(\theta) \leq x(\theta) \text{ for all } \theta \in [a,b), \\ x(b^-) & \text{if } x_R(\theta) \geq x(\theta) \text{ for all } \theta \in [a,b), \\ \max\{x(t^-), x_R(t^+)\} & \text{otherwise,} \end{cases}$$

where $t := \sup\{\theta \in (a,b) : x_R(\theta) \geq x(\theta)\}$. Note x^* is increasing and pointwise between x_R and x, so $V[x^*] \geq V[x]$ by Lemma 1. □

Lemma 3. *Suppose x_R is increasing on $[a,b)$. Then any increasing allocation rule x can be weakly improved upon by:*

$$x^*(\theta) = \begin{cases} \min\{x(b), \max\{x(a), x_R(\theta)\}\}, & \theta \in [a,b), \\ x(\theta), & \theta \notin [a,b). \end{cases}$$

Proof. Fix any increasing x. Note x^* is pointwise between x and x_R, so $V[x^*] \geq V[x]$ by Lemma 1. Moreover, x^* is increasing because x_R is increasing on $[a,b)$ by assumption. □

Observations similar to Lemmas 2 and 3 were made by Sandmann (2022) who studies the optimality of sparse menus in a price discrimination problem. Given the above lemmas, the proof of Theorem 1 is straightforward. By Assumption 2, $[0,1)$ can be partitioned into finitely many intervals $[i_n, i_{n+1})$ where x_R is monotonic. Then, by Lemmas 2 and 3, if the planner's problem has a solution, it has one of the form in (2) with $v_1 \leq v_2 \leq \cdots \leq v_{|\mathcal{I}|-1}$. Solving problem (I) recovers this solution.

5 Solution algorithm

I now provide a simple algorithm that solves the planner's problem under the following additional assumption:

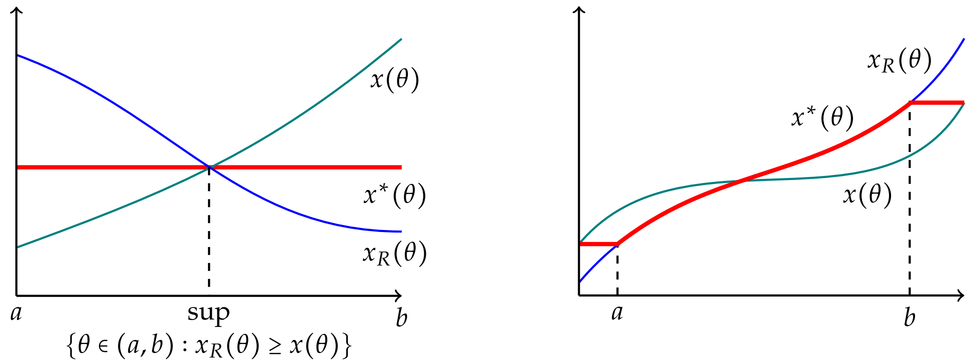

Figure 1a: Constructing an improvement in the proof of Lemma 2.

Figure 1b: Constructing an improvement in the proof of Lemma 3.

Assumption 3. *The virtual value satisfies the following properties:*

1. *The planner chooses allocations from a closed interval:* $\mathcal{X} = [l, h]$.
2. $J(\cdot, \theta)$ *is continuous and weakly concave for every* $\theta \in [0, 1]$.

The algorithm uses the following transformation $\mathrm{T} : [l,h]^{[0,1]} \times [l,h] \times \mathbb{N} \to [l,h]^{[0,1]}$:

$$\mathrm{T}[x, \tilde{v}, n](\theta) := \begin{cases} \min\{\tilde{v}, x(\theta)\} & \text{if } \theta < i_n, \\ \tilde{v} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\{\tilde{v}, x(\theta)\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \\ h & \text{if } \theta = i_{n+1}, \\ x(\theta) & \text{if } \theta > i_{n+1}. \end{cases}$$

When $\mathrm{T}[\cdot, \tilde{v}, n]$ is applied to x, the allocation rule is truncated from above by $\tilde{v}$ before the nth critical point and set equal to $\tilde{v}$ or truncated by it from below between the nth and $(n+1)$st critical points. It is also set to h at the $(n+1)$st critical point. The following algorithm takes in the solution to the relaxed problem, x_R, and the set of critical points $\mathcal{I}$ as inputs and iteratively applies this transformation to produce a solution to the planner's problem:

Algorithm 1

$r \leftarrow x_R$
$n \leftarrow 1$
while $n \leq |\mathcal{I}| - 1$ **do**
    $v^* \leftarrow \text{argmax}_{\tilde{v} \in [l,h]} \int_0^{i_{n+1}} J(\mathrm{T}[r, \tilde{v}, n](\theta), \theta) d\theta$
    $r \leftarrow \mathrm{T}[r, v^*, n]$
    $n \leftarrow n + 1$
end while
return r

Intuitively, each iteration of Algorithm 1 starts with the optimum subject to monotonicity from 0 until the nth critical point and "irons out" this partial solution further to produce a solution subject to monotonicity until the $(n+1)$st critical point (Figure 2).

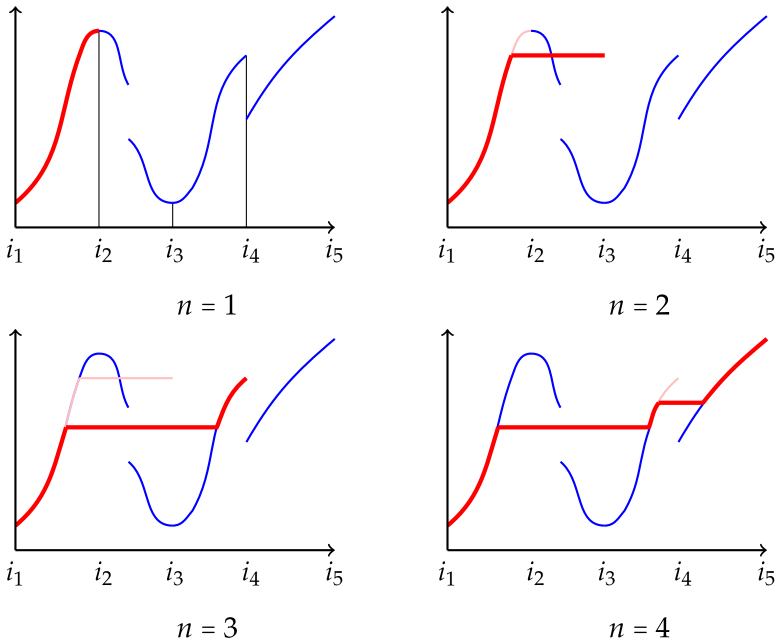

Figure 2: Algorithm 1 recursively transforming x_R (blue) into subsequent $\mathrm{T}[x, v^*, n]$ (red).

Theorem 2. *Under Assumption 3, the output of Algorithm 1 solves the planner's problem.*

Proof of Theorem 2. For $a \in [0,1)$, define the partial objective:

$$F_a[x] := \int_0^a J(x(\theta), \theta) d\theta.$$

Now, consider the set of allocation rules that are increasing on $[0,a)$ and below k everywhere on that interval. Let $\mathcal{O}_a^k$ contain the allocation rules that maximize F_a over this set:

$$\mathcal{O}_a^k := \arg\max \Big\{ F_a[x] \ \text{ s.t. } \ x : [0,1] \to [l,h] \text{ is increasing on } [0,a), \ x \leq k \text{ on } [0,a) \Big\}.$$

I first show $\mathcal{O}_a^k$ is non-empty for all $k \in [l,h]$ and $a \in (0,1]$. Let $\mathcal{A}_a^k$ be the set of increasing functions $x : [0,a] \to [l,k]$. By Helly's selection theorem, $\mathcal{A}_a^k$ is sequentially compact under pointwise convergence. If $x_m \to x$ pointwise in $\mathcal{A}_a^k$, then $J(x_m(\theta), \theta) \to J(x(\theta), \theta)$ for each θ. By Assumption 1, $|J(x_m(\theta), \theta)| \leq M$ uniformly, so dominated convergence gives $F_a[x_m] \to F_a[x]$. Hence F_a attains its maximum on $\mathcal{A}_a^k$.

I also prove the following proposition. It says that when we have an x in $\mathcal{O}_a^h$ and impose the additional constraint that x be below k on $[0,a)$, we need not resolve the problem, but can simply truncate the solution without this constraint by k.

Proposition 1. *If $x_a \in \mathcal{O}_a^h$, then $\min\{k, x_a\} \in \mathcal{O}_a^k$.*

Proof. Fix x_a and note $\min\{x_a(\theta), k\}$ is admissible in the problem defining $\mathcal{O}_a^k$. Fix any other admissible x and define:

$$\tilde{x}(\theta) = \begin{cases} x(\theta) + \max\{x_a(\theta) - k, 0\}, & \theta \in [0,a), \\ h, & \text{otherwise.} \end{cases}$$

Since x and $\max\{x_a(\theta) - k,\ 0\}$ are non-decreasing, so is $\tilde{x}$. Moreover, $x(\theta) \le k$ for $\theta < a$ and $x_a(\theta) \le h$, so $\tilde{x}(\theta) \le h$. Now, $x_a \in \mathcal{O}_a^h$, and so:

$$\int_0^a J(x_a(\theta), \theta)\, d\theta \ \ge\ \int_0^a J(\tilde{x}(\theta), \theta)\, d\theta. \tag{3}$$

Fix θ and consider two cases. If $x_a(\theta) \le k$, then $\min\{x_a(\theta), k\} = x_a(\theta)$ and $\tilde{x}(\theta) = x(\theta)$, hence:

$$J(\min\{x_a(\theta), k\}, \theta) - J(x(\theta), \theta) = J(x_a(\theta), \theta) - J(\tilde{x}(\theta), \theta). \tag{4}$$

Now suppose $x_a(\theta) > k$. Let $\delta := k - x(\theta) \ge 0$ and note that:

$$\min\{x_a(\theta), k\} = x(\theta) + \delta.$$

Since $J(\cdot, \theta)$ is concave, for all u, u' such that $u < u'$:

$$J(u + \delta, \theta) - J(u, \theta)\ \ge J(u' + \delta, \theta) - J(u', \theta).$$

Recall $x(\theta) < \tilde{x}(\theta)$, so we have:

$$J(x(\theta) + \delta, \theta) - J(x(\theta), \theta)\ \ge\ J(\tilde{x}(\theta) + \delta, \theta) - J(\tilde{x}(\theta), \theta).$$

Note $x(\theta) + \delta = k = \min\{x_a(\theta), k\}$ and $\tilde{x}(\theta) + \delta = x(\theta) + x_a(\theta) - k + (k - x(\theta)) = x_a(\theta)$, so:

$$J(\min\{x_a(\theta), k\}, \theta) - J(x(\theta), \theta)\ \ge\ J(x_a(\theta), \theta) - J(\tilde{x}(\theta), \theta). \tag{5}$$

Integrating over $\theta \in [0, a]$ and combining (3), (4) and (5) yields:

$$\int_0^a \big[J(\min\{x_a(\theta), k\}, \theta) - J(x(\theta), \theta)\big]\, d\theta\ \ge\ \int_0^a \big[J(x_a(\theta), \theta) - J(\tilde{x}(\theta), \theta)\big]\, d\theta\ \ge\ 0,$$

giving $\int_0^a J(\min\{x_a(\theta), k\}, \theta)\, d\theta \ge \int_0^a J(x(\theta), \theta)\, d\theta$. □

I now present an inductive proof of Theorem 2. Recall that $|\mathcal{I}| \ge 2$ since $0, 1 \in \mathcal{I}$. The base case demonstrates that the first iteration of the algorithm produces $r \in \mathcal{O}_{i_2}^h$. The step shows that when the nth iteration starts with $r \in \mathcal{O}_{i_n}^h$, it produces $r \in \mathcal{O}_{i_{n+1}}^h$. The base and the step thus imply that the algorithm will produce $r \in \mathcal{O}_{i_{|\mathcal{I}|}}^h$ solving the planner's problem in $|\mathcal{I}| - 1$ steps.

Base. Since $\mathcal{O}_{i_2}^h \neq \varnothing$, Lemmas 2 and 3 tell us there exists $x^* \in \mathcal{O}_{i_2}^h$ for which:

$$x^*(\theta) := \begin{cases} v_1 & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is decreasing,} \\ \min\{v_2, \max\{v_1, x_R(\theta)\}\} & \text{if } \theta \in [i_1, i_2) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, by Lemma 1 we can without loss set $v_2 = h$. The first iteration of the algorithm will recover such an x^* when optimizing over $\tilde{v}$.

Step. Since $\mathcal{O}^h_{i_{n+1}} \neq \varnothing$, Lemmas 2 and 3 tell us there exists $x^* \in \mathcal{O}^h_{i_{n+1}}$ for which:

$$x^*(\theta) := \begin{cases} v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \min\{v_{n+1}, \max\{v_n, x_R(\theta)\}\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing,} \end{cases}$$

where $x^*(i_n) = v_n$. Moreover, by Lemma 1 we can without loss set $v_{n+1} = h$. Let $r \in \mathcal{O}^h_{i_n}$ be the allocation rule produced by the $n-1$st iteration of the algorithm. By Proposition 1, $\min\{v_n, r\} \in \mathcal{O}^{v_n}_{i_n}$. Since $x^*(\theta) \leq v_n$ for $\theta < i_n$, this implies $F_{i_n}[\min\{v_n, r\}] \geq F_{i_n}[x^*]$ and so the following allocation rule weakly improves upon x^*:

$$x^{**}(\theta) := \begin{cases} \min\{v_n, r(\theta)\} & \text{if } \theta < i_n, \\ v_n & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is decreasing,} \\ \max\{v_n, x_R(\theta)\} & \text{if } \theta \in [i_n, i_{n+1}) \text{ where } x_R \text{ is increasing.} \end{cases}$$

Moreover, x^{**} is increasing on $[0, i_{n+1})$ and takes values in $[l, h]$, so x^{**} belongs to $\mathcal{O}^h_{i_{n+1}}$. The nth iteration of the algorithm will recover such an x^{**} when optimizing over $\tilde{v}$.

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