# Screening with damages and ordeals*

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**Abstract**

A welfare-maximizing designer allocates two kinds of goods using two wasteful screening instruments: *ordeals*, which enter agents' utilities additively, and *damages*, which harm agents in proportion to their values for the goods. If agents have common valuations for one of the goods, damages always lead to Pareto-dominated mechanisms: any allocation using damages can be implemented with ordeals alone, while leaving higher rents to inframarginal types. However, damages can be optimal when agents' valuations for both goods are heterogeneous: with multidimensional types, the two devices differ in how they sort agents into the available options, and optimal sorting can sometimes require damages. I nevertheless identify distributional conditions under which damages are not optimal. In those cases, the optimal mechanism produces an efficient allocation by posting "market-clearing" ordeals for each good.

## 1 Introduction

Goods provided through public programs are often less desirable than their private-market counterparts. Public housing projects are frequently located in disadvantaged or peripheral neighborhoods. Medicaid covers fewer provider options than private insurance plans. Food assistance delivered through in-kind or voucher programs such as SNAP restricts choice and, in the latter case, forces households to keep track of eligible products. While these restrictions may be costly for recipients, they can also serve as self-targeting devices (Nichols and Zeckhauser, 1982; Besley and Coate, 1991; Currie and Gahvari, 2008). For example, if subsidized housing is mostly built in poorer or less desirable neighborhoods, families with better alternatives may decide not to apply and instead remain in the private market. In this way, the subsidy implicit in the program disproportionately benefits poorer people. Similar arguments have been made for other screening devices, such as queues or work requirements. However,

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quality reductions and usage restrictions have an important screening property that these latter instruments do not: they are *more costly* for households whose values for the allocated goods are higher. Indeed, a sicker patient on government health insurance is more likely to be burdened by her plan's network or usage restrictions than a healthier patient. Likewise, a family relying more heavily on food vouchers will find the program's constraints more stringent because it spends less of its own money on food. By contrast, screening devices such as frequent recertification requirements impose disutilities that are *separable* from recipients' values for the allocated good.[1]

I compare the properties of these two kinds of screening instruments. I call the former class of instruments, whose costs to agents increase with their values for the good, *damages*. I refer to the latter class, whose costs are separable from agents' values, as *ordeals*. Both classes of instruments are common in practice. In particular, damages can be understood more broadly than literal quality reductions: for example, delays in allocation are a prominent form of damage. Consider affordable housing, which is frequently allocated through waitlists. Each period during which a household waits deprives it of the flow value of the apartment, making the cost of delay *multiplicative* in its value for the unit.[2] It is worth noting, however, that the screening properties of delays (and thus waitlists) are different from those of *queues*, à la Nichols et al. (1971), where agents waste time standing in line. Since queueing does not affect the value derived from the good, under my definition, it is an *ordeal*. Other examples of wasteful screening devices that are ordeals include form-filling and bureaucratic processes, work requirements, and traveling to a distant office (Nichols and Zeckhauser, 1982; Besley and Coate, 1992; Kleven and Kopczuk, 2011; Dupas et al., 2016; Alatas et al., 2016; Deshpande and Li, 2019).

In my model, a designer uses these screening devices to allocate two kinds of goods to agents who have distinct values for each of them. Considering such heterogeneity lets me study not only which types of agents are *screened out* by damages and ordeals, but also how these instruments can be used to "sort" agents into the available options. Good heterogeneity is also an important feature of settings to which my model applies: affordable housing programs, for instance, offer units that vary in location and size, with households' preferences over them having a strong horizontal component (Waldinger, 2021). Moreover, these apartments are often allocated via waitlists, where wait-times differ substantially between developments. Participating households therefore trade off their values for specific units against expected wait-time (Van Ommeren and Van der Vlist, 2016).[3] Thus, in public housing programs, wait-times—a damage instrument—largely assume the role of prices, balancing supply and demand and "sorting" participants into the available units. Healthcare provides another example of a set-

[1]The relationship between the costs of screening instruments and values for the allocated good discussed here is "mechanical," in the sense that it is built into the utility specification. Additional dependence arising from correlation between values and costs is introduced in Section 7.2.

[2]For flow goods that can be enjoyed in every period after receipt, delays deprive recipients of periods of use. For consumable goods, delays lead to temporal discounting of the good's value.

[3]While affordable housing programs around the world differ in terms of the exact mechanism or its implementation, the same core trade-off is often present.

ting where the heterogeneity of allocated goods is first-order: many conditions admit both low- and high-intensity therapies, for example, musculoskeletal injuries may be treated with physical therapy or surgery. Because the supply of certain treatments is limited, patients who seek scarce or costly procedures are often required to obtain referrals or to document that less intensive treatments have been tried first. Indeed, World Health Organization (2023) notes that referral systems are intended to "ensure patient access to specialist healthcare when needed, while maintaining resource efficiency." Such requirements may reduce pressure on the system by ensuring that only higher-need patients pursue the most resource-intensive options, but they also delay the therapy itself, acting as a *damage*.

I therefore consider a mechanism design setting where a welfare-maximizing designer allocates fixed supplies of goods to agents. She chooses deterministic mechanisms that combine two wasteful screening instruments: ordeals and damages to the allocated goods. My main result (Theorem 1) identifies distributional conditions under which the optimal mechanism does not damage either good. In those cases, the welfare-maximizing mechanism posts "market-clearing" ordeals for each good, allocating the entire supply efficiently. However, I also show that damages *can* be optimal when these distributional conditions are violated. I then study the case where agents differ in their values for a single type of scarce good and have common values for an unlimited outside option. In such cases, a stronger result holds: damages always lead to Pareto-dominated mechanisms. Intuitively, damaging the good is less efficient at screening out low-value agents, as it is *more costly* to high-value inframarginal recipients than it is to the marginal recipient. Since ordeals are equally costly to everyone, they could be used to produce the same allocation while preserving more rents for inframarginal recipients. This result points to a contrast: when agent heterogeneity is one-dimensional, using damages is never optimal. When agents differ with respect to their values for two goods, however, the designer can sometimes benefit from damaging them. I provide an intuition for this difference: in the case of one-dimensional heterogeneity, any implementable way of "sorting" agents into the two options can be implemented using only ordeals. The designer can therefore always implement the best sorting pattern using ordeals, which are less harmful to inframarginal types. This is no longer the case, however, when types are multidimensional: in that case, damages and ordeals "sort" agents into goods in qualitatively different ways, and optimal sorting patterns can sometimes be implemented only by using damages.

The core of my paper studies a stylized model illustrating the properties of the two classes of screening instruments where damaging goods does not produce any direct benefit for the designer. However, I subsequently consider extensions that show that my insights and methods survive when other practically important effects are also considered. I first relax the assumption that both screening devices are fully wasteful. This may, for instance, capture the case where ordeals are interpreted as monetary payments or reductions in subsidies, which generate revenue for the designer. In Section 7.1, I therefore extend my analysis to the case where ordeals are only *partially* wasteful, and show this does not affect Theorem 1. Second, I consider the case where agents differ in how costly they find the ordeal. For instance, when screening is done with monetary payments, poorer agents whom the program tries to target may find them more burdensome. Conversely, the same agents may be more willing to wait, travel or endure other inconveniences in order to get the good (Dupas et al., 2016). I show that my solution method directly extends to the case of heterogeneous ordeal costs and I provide modified statistical conditions under which damages are suboptimal in this setting. I also consider the case where damaging goods can generate savings for the designer. In this case, I find conditions under which the optimal mechanism is simple and can be implemented using a three-option menu. Lastly, in Appendix A, I draw a connection between my baseline model and waitlist mechanisms described in the introduction. Importantly, in this environment wait-times are not a direct design choice but arise naturally as byproducts of the system's equilibrium dynamics. Nonetheless, I provide a microfoundation which shows that the steady state of such a system maps onto my baseline model.

I conclude the analysis by discussing its implications for the design of social programs, and in particular for public housing allocation. There, my results suggest that large imbalances in the lengths of different waitlists, which effectively correspond to screening using damages, may be undesirable. The policymaker should instead consider rebalancing the prices or subsidies between different housing developments in a way that brings their wait-times closer together.

From a technical perspective, my model is an instance of a tractable multidimensional screening problem. By restricting attention to deterministic mechanisms, I am able to characterize them as pairs of ordeal and damage menus for the two goods. This in turn allows me to represent two-dimensional mechanisms as (endogenously) interconnected single-dimensional screening problems. The interaction between them is summarized by a boundary in the type space that separates the sets of types who choose each good. The multidimensional problem can then be broken up into two stages: first, determining the optimal way to implement a given boundary, and second, solving an optimal control problem to select the best boundary among all implementable ones. While my paper applies this method to the problem of a welfarist designer, similar ideas could be useful for studying other settings, such as the problem of a two-good monopolist choosing deterministic mechanisms for selling to unit-demand consumers.

Apart from the aforementioned work on screening with ordeals in social programs, this paper relates closely to a more general literature on using costly screening and money-burning to maximize welfare (Hartline and Roughgarden, 2008; Condorelli, 2012). A related literature also studies the use of wasteful screening devices by a monopolist, with the seminal paper by Deneckere and McAfee (1996) establishing conditions under which the seller would want to damage goods. Yang (2021) considers a more general problem where a monopolist has access to both wasteful and non-wasteful instruments, and characterizes cases where the wasteful one should not be used. However, this literature focuses on the problems of allocating a single good. By considering a model with heterogeneous goods, I explore an additional role of wasteful screening devices, where they are also used to "sort" agents into the available options. Indeed, my results show that while damages can never be optimal in the single-good case, the designer may find them useful when two kinds of goods are allocated. My comparison of the screening properties of damages and ordeals also relates to the work of Akbarpour et al. (2023) who ask when one screening device dominates another for a planner aiming to maximize a social welfare function. Unlike them, I allow the designer to *combine* instruments and show that, under certain distributional conditions, screening with ordeals alone dominates any mechanism using both devices. Finally, my paper relates to a literature on waitlist design. While no paper has studied combining waitlists with payments or ordeals in settings with heterogeneous goods, a substantial literature examines designing such waitlists without transfers. Arnosti and Shi (2020) and Waldinger (2021) study the effects of restricting recipients' choice on targeting. Barzel (1974), Bloch and Cantala (2017), and Leshno (2022) observe that in environments with homogeneous waiting costs, wait-times may "act as prices," screening for agents with higher valuations. I refine this intuition by showing that the screening properties of wait-times are different when the cost of waiting stems from delayed receipt.

## 2 Model

A designer distributes two types of goods, $A$ and $B$. Their supplies are equal to $s_A, s_B > 0$. There is a unit mass of agents whose values for the two goods are given by $a$ and $b$, respectively. Agents' values $(a,b)$ are distributed according to $F$ with full support on $[0,1]^2$. The designer chooses a menu of qualities and ordeals for each of the goods. That is, an agent can choose which good she wants to get and then choose a quality and ordeal option from the relevant good's menu. She can also choose not to participate, which gives her utility 0. When a type-$(a,b)$ agent participates and receives good $y$, her utility is:

$$x \cdot a - c \;\text{ if }\; y = A,$$

$$x \cdot b - c \;\text{ if }\; y = B,$$

where $c \in \mathbb{R}_+$ is the ordeal the agent completes and $x \in [0,1]$ is her good's quality. Whenever $x < 1$, we say the good has been *damaged*. The designer chooses the menus to maximize the total welfare of agents. We can reduce her problem to picking allocation rules for ordeals, $c : [0,1]^2 \to \mathbb{R}_+$, qualities, $x : [0,1]^2 \to [0,1]$, and goods, $y : [0,1]^2 \to \{\varnothing, A, B\}$, to maximize welfare:[4]

$$\int u_{a,b}(a,b)\, dF(a,b), \tag{W}$$

subject to (IC) and (IR) constraints, and the supply constraint (S):

$$\text{for all } (a,b),(a',b') \in [0,1]^2, \quad u_{a,b}(a,b) \geq u_{a,b}(a',b'), \tag{IC}$$

$$\text{for all } (a,b) \in [0,1]^2, \quad u_{a,b}(a,b) \geq 0, \tag{IR}$$

[4]While the standard Revelation Principle fails for deterministic mechanisms, I show in Appendix B that this direct-revelation formulation is indeed valid.

$$\int \mathbb{1}_{y(a,b)=A}\, dF(a,b) \leq s_A, \quad \int \mathbb{1}_{y(a,b)=B}\, dF(a,b) \leq s_B. \tag{S}$$

Here $u_{a,b}(a',b')$ denotes the utility type $(a,b)$ gets from reporting $(a',b')$ in the mechanism $(c,x,y)$. I call a mechanism $(c,x,y)$ satisfying (IC),(IR), (S) *feasible*.

**Remark 1.** *There are several ways in which my model departs from the standard quasi-linear multidimensional screening setting studied, among others, by Rochet and Choné (1998), Manelli and Vincent (2006), and Daskalakis, Deckelbaum, and Tzamos (2017). First, I consider a designer maximizing welfare, rather than revenue. Second, I consider only deterministic mechanisms; together with the assumption of unit demand, this means that every agent is assigned at most one kind of good.*

*The restriction to deterministic mechanisms reflects practical constraints and considerations present in many settings. For instance, lotteries for public housing may be perceived as unfair, with some cities eschewing them in favor of deterministic waitlists for this reason.*[5] *Furthermore, housing lotteries sometimes raise concerns over corruption and draw-faking, prompting calls to replace them with more transparent mechanisms like first-come-first-served waitlists.*[6] [7] *Randomization may also be impractical in other settings; for instance, a socialized healthcare program might be unwilling to randomize patients into therapies and might opt for a waitlist-based mechanism instead.*

*Restricting attention to deterministic mechanisms is also advantageous from a modelling perspective. First, it allows for a clearer comparison of the properties of damages and ordeals, which is the focus of this paper. It also makes an otherwise unwieldy setting tractable—while disallowing lotteries means that the designer's problem is no longer a linear program, it also imposes additional structure which often leads to simple and interpretable solutions.*

**Remark 2.** *From the agent's perspective, getting a damaged good with quality $x < 1$ would be identical to receiving it with probability $x$. However, randomization and damages differ from the perspective of the designer, as they enter the supply constraint differently: allocating a good with probability $x$ takes up only $x$ of the supply, while damaging it to quality $x$ still uses up a whole unit. In this sense, damages are strictly inferior to randomization. Nevertheless, I later show that the designer may still want to use them if randomization is not allowed.*

## 3 When are damages not optimal?

My main result specifies conditions under which the optimal mechanism does not damage any goods. The first condition imposes a regularity requirement on the type distribution:

[5] For instance, Whistler, Canada "allocates units based on a waitlist, a method chosen due to its perceived fairness and ease of administration, though lottery and points schemes have been used in the past" (City of Vancouver, 2016).

[6] https://www.camara.leg.br/noticias/523091-projeto-veda-sorteio-na-selecao-dos-beneficiarios-do-minha-casa-minha-vida/

[7] https://citymeetings.nyc/meetings/new-york-city-council/2025-04-29-1000-am-committee-on-housing-and-buildings/chapter/consideration-of-moving-from-lottery-system-to-universal-waiting-list-for-affordable-housing

**Assumption 1.** *The value distribution $F$ has a Lipschitz continuous density $f$. Moreover, the inverse anti-hazard rates of the conditional distributions,*

$$\frac{F_{A|B}(a|b)}{f_{A|B}(a|b)}, \quad \frac{F_{B|A}(b|a)}{f_{B|A}(b|a)},$$

*are increasing in $a$ and $b$, and for each ratio at least one of the two monotonicities is strict.*

To understand the anti-hazard-rate condition, fix $b$ and consider the distribution of the value for good $A$ conditional on the value for good $B$. The ratio

$$\frac{F_{A|B}(a \mid b)}{f_{A|B}(a \mid b)} \tag{1}$$

then compares the probability mass below $a$ to the density at $a$. The condition requires the accumulated mass below $a$ to increase relative to the density at $a$, or equivalently requires the conditional cdf to be log-concave. In particular, this means that the probability mass below $a$ has to accumulate smoothly, which rules out sharp spikes in the conditional density. The second part of the assumption says that conditioning on higher $b$ also makes the mass below $a$ larger relative to the density at $a$. In particular, this latter requirement rules out the case where the two variables are *strictly positively affiliated*. Intuitively, if the values are related positively, conditioning on a higher $b$ makes $a$ look more like it came from further in the left tail, which makes the ratio (1) fall. The condition does, however, allow for *negative* dependence between the two dimensions.[8]

In the special case where the values for the two goods are independent, $F(a,b) = F_A(a)\, F_B(b)$, the cross-effect is absent, and the condition reduces to a one-dimensional shape restriction on the marginals. It then simply requires that

$$\frac{F_A(a)}{f_A(a)}, \quad \frac{F_B(b)}{f_B(b)},$$

be increasing, with one of them strictly so.

It is worth noting that Assumption 1 is not strictly necessary for Theorem 1. While proving this is difficult, I expect the result holds under less stringent conditions.

The second condition is purely technical and assumed for analytical convenience; it imposes

[8] Indeed, when $f$ is differentiable, we have:

$$\frac{d}{db}\frac{F_{A|B}(a \mid b)}{f_{A|B}(a \mid b)} = \frac{1}{f(a,b)} \int_0^a f(t,b) \left( \frac{\partial}{\partial b} \log f(t,b) - \frac{\partial}{\partial b} \log f(a,b) \right) dt,$$

so the sign of the derivative depends on how $\frac{\partial}{\partial b} \log f(a,b)$ varies with $a$. If $\frac{\partial^2}{\partial a \partial b} \log f(a,b) \leq 0$, the integrand is always positive. If instead $\frac{\partial^2}{\partial a \partial b} \log f(a,b) \geq 0$, i.e. the two dimensions are affiliated, it is negative.

the following restriction on the space of admissible mechanisms:[9]

**Assumption 2.** *The designer is restricted to quality rules $x : [0,1]^2 \to [0,1]$ that are piecewise continuously differentiable in each dimension of the type.*[10]

Under these assumptions, the designer's optimal mechanism is given by the following result:

**Theorem 1.** *Suppose Assumptions 1 and 2 hold and that there are fewer goods than agents: $s_A + s_B \leq 1$. Then the optimal mechanism offers only two options: it allocates undamaged goods A and B with ordeals $c_A$ and $c_B$, respectively. These ordeals are chosen so that the whole supply of both goods is allocated.*

This optimal mechanism has a simple interpretation: the designer posts a pair of "market-clearing" ordeals which act as prices, replicating the competitive equilibrium allocation of goods. The allocation under this mechanism is illustrated in Figure 1: agents whose values for both goods are sufficiently low get nothing. The other agents choose one of the goods and get an undamaged version of it after completing the ordeal $c_A$ or $c_B$.

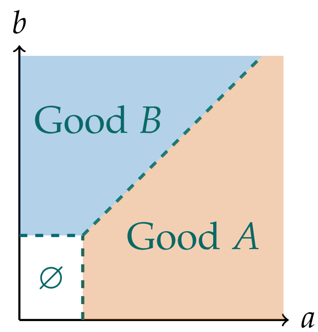


Figure 1: Optimal allocation under the conditions of Theorem 1.

While it is easy to see that this mechanism is optimal when transfers are welfare-neutral, the theorem says the same is true when the "revenue" of the mechanism is completely wasted.

## 4 Proof of Theorem 1

In this section, I outline the core steps and intuitions behind the proof of the result, as well as explain the role of both assumptions. Results referenced throughout this section are shown in the appendix.

[9]Indeed, I conjecture this restriction is not binding.

[10]Note that Assumption 2 allows for allocation rules that are discontinuous at finitely many points.

**Boundary structure of mechanisms.** I now show that implementable mechanisms can be conveniently represented using a boundary in the type space. First, however, note that we can restrict attention to mechanisms that allocate positive masses of both goods.[11] Let us then describe the properties of such mechanisms. It will be convenient to characterize them in terms of $A, B$-indirect utilities $U_A, U_B : [0,1] \to \mathbb{R}_+$, defined as follows:

$$U_A(a) = \max_{\substack{(a',b') \\ y(a',b')=A}} \left(x(a',b')\,a - c(a',b')\right)_+, \qquad U_B(b) = \max_{\substack{(a',b') \\ y(a',b')=B}} \left(x(a',b')\,b - c(a',b')\right)_+.$$

Intuitively, $U_A(a)$ and $U_B(b)$ represent the highest utility type $(a,b)$ could get from selecting some quality and ordeal option for the $A$- and the $B$-goods, respectively (or not participating). Then agents for whom $U_A(a) > U_B(b)$ choose good $A$ and those for whom $U_A(a) < U_B(b)$ choose good $B$. Note that $U_A$ and $U_B$ are convex and increasing, and that they depend only on one dimension of the type—an agent's value for good $B$ does not affect her choice of quality and ordeal option if she chooses good $A$.

**Definition 1.** *Define a mechanism's **lowest participating values** as follows:*

$$\underline{a} = \sup\{a : \ U_A(a) = 0\}, \quad \underline{b} = \sup\{b : \ U_B(b) = 0\}.$$

*Let a **boundary** be a strictly increasing, continuous function $z : [\underline{a}, \overline{a}] \to [\underline{b}, \overline{b}]$ such that $\overline{a} \leq 1$ and $\overline{b} \leq 1$, with one of them holding with equality.*[12]

**Proposition 1.** *Agents' choices of goods are characterized by the mechanism's lowest participating values $\underline{a}, \underline{b}$ and a boundary $z$:*

1. *Types $(a,b) < (\underline{a}, \underline{b})$ do not get either good.*[13]
2. *Types for whom $a > \underline{a}$ and $b < \underline{b}$ get good $A$; types for whom $a < \underline{a}$ and $b > \underline{b}$ get good $B$.*
3. *Types $(a,b) > (\underline{a}, \underline{b})$ get good $A$ if $(a,b)$ is below the boundary $z$, that is, if $z(a) > b$, and get good $B$ if $(a,b)$ is above the boundary $z$, that is, if $z(a) < b$.*

*Moreover, types on the boundary are indifferent between their favorite options for both goods, thus:*

$$U_A(a) = U_B(z(a)) \quad \textit{for all} \quad a \in [\underline{a}, \overline{a}]. \tag{2}$$

[11] Indeed, consider a mechanism allocating only $A$. Now, augment it by adding to its menu of options one which allocates good $B$ with $x = 1$ with an ordeal of $c = 1 - \epsilon$. Since the distribution $F$ of values $(a,b)$ is full-support, for any $\epsilon > 0$ there will be a mass of agents who prefer the $B$-good option to any $A$-good option offered; that is, introducing this option increases their welfare. Note that adding the $B$-option only relaxes the supply constraint on $A$, and that the mass of the agents taking the $B$-option converges to 0 as $\epsilon \to 0$. Thus, the augmented mechanism satisfies the supply constraint for $B$ for $\epsilon$ small enough.

[12] Note that despite the boundary $z(a)$ being written as a function of $a$, the characterization in the following lemma could equally well be formulated in terms of $z^{-1}(b)$.

[13] When comparing vectors, I use $\geq$ and $>$ for pointwise comparisons.

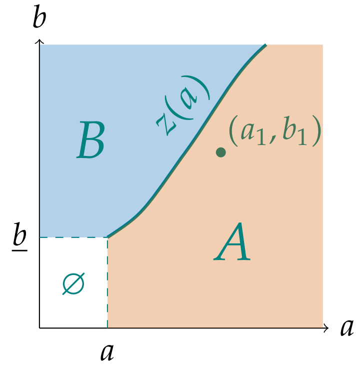


Figure 2: Types below the boundary (orange) choose good $A$ and types above it (blue) choose good $B$.

When all qualities of $A$- and $B$-goods come with ordeals, types with sufficiently low values for both of them, i.e. $(a,b) < (\underline{a},\underline{b})$, do not participate. To understand the good choices of participating types, consider some $(a_1, b_1)$ choosing good $A$ (Figure 2). Then any type $(a,b)$ with $a > a_1$ and $b < b_1$ will also choose good $A$—since she values the $A$-good even more than $(a_1, b_1)$ and values the $B$-good even less, all the ordeal and quality options for good $B$ are strictly less attractive to her than they were to $(a_1, b_1)$. We can now notice that the types who are indifferent between their best options for either good lie on an increasing curve $z$ originating from $(\underline{a},\underline{b})$. By the above logic, all types below this curve choose $A$ and pick some ordeal and quality option from its menu, while types above it choose $B$.

We can therefore think of our multidimensional mechanism design problem as two single-dimensional problems connected endogenously through the boundary $z$. While agents on either side effectively face one-dimensional problems, making one of the goods more attractive invites more types to switch to it, effectively deforming the boundary.

Moreover, by the definition of a boundary, at least one endpoint lies on the boundary of the unit square: either $\overline{a} = 1$ (the boundary reaches the right edge first) or $\overline{b} = 1$ (it reaches the top edge first). If $\overline{b} = 1$, it is convenient to parametrize the boundary as a function of $a$, i.e. $z(a)$ on $[\underline{a}, \overline{a}]$. If instead $\overline{a} = 1$, we can equivalently work with the inverse parametrization, $z^{-1}(b)$ on $[\underline{b}, \overline{b}]$. Since the two cases are symmetric, I assume without loss that $\overline{b} = 1$.

**Reformulating the objective.** We can use the boundary $z$ and the $A$-indirect utility $U_A$ to rewrite total welfare in a concise way. To that end, we define an *extended boundary* $\hat{z}$ which is constructed from $z$ as follows:

$$\hat{z}(a) = \begin{cases} 0, & \text{if } a < \underline{a}, \\ z(a), & \text{if } a \in [\underline{a}, \overline{a}], \\ 1, & \text{if } a > \overline{a}. \end{cases}$$

That is, $\hat{z}$ equals $z$ on the latter's domain, takes value zero below it and takes value 1 above it (Figure 3). Throughout, I use $\hat{z}^{-1}$ to denote its generalized inverse.

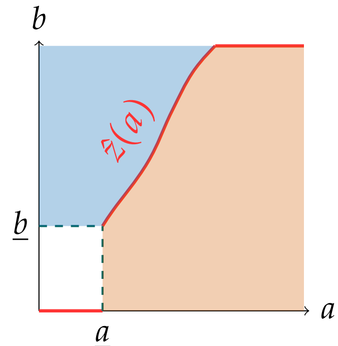


Figure 3: Extended boundary $\hat{z}$.

**Lemma 1.** *Consider a mechanism with a boundary $z$ and $A$-indirect utility $U_A$. Total welfare under this mechanism is then given by:*

$$U_A(1) - \int_0^1 U_A'(a) \cdot F(a, \hat{z}(a)) da. \tag{3}$$

Intuitively, the lemma shows how the welfare of all agents in the mechanism can be written using only the welfare of agents choosing good $A$ and the shape of the boundary. To see why this is the case, note that the utilities of agents getting good $A$ depend only on their values for good $A$ and not on those for good $B$, and vice versa. Moreover, recall that the agents on the boundary are indifferent between the two goods. It therefore follows that types lying on the same inverted L-shaped curves in Figure 4 all have the same utility. Indeed, this observation is captured in the equation defining the boundary:

$$U_A(a) = U_B(z(a)) \quad \text{for all} \quad a \in [\underline{a}, \overline{a}]. \tag{4}$$

We can therefore calculate welfare by integrating over types who choose $A$ while also taking into account the $B$-taking types on the same $L$-shaped curves. Such a calculation yields the expression (3). It is worth noting that this representation of total welfare does not rely on the anti-hazard rate and piecewise continuous differentiability assumptions made for Theorem 1.

This form of the objective also bears a resemblance to a Myersonian virtual value which would appear in a single-dimensional setting with a welfarist objective (see e.g. Condorelli (2012)). Moreover, by the envelope theorem, we can think of $U_A'(a)$ as the quality allocated to agents who receive good $A$ and value it at $a$. Unfortunately, however, the setting does not lend itself to a Myersonian solution method. The reason for this is two-fold. First, the supply constraint in my model is not on total quality, but on the *mass of agents who receive good A*; this constraint depends not on the total assignment of $U_A'$, but on the probability masses on both sides of

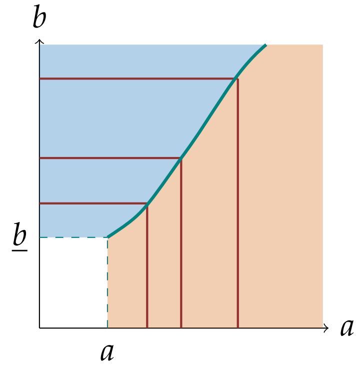


Figure 4: Agents on the same $L$-shaped curves have equal utilities.

the boundary $z$. Second, the shape of the "virtual value" itself is endogenous to the choice of boundary. Indeed, it is optimizing over the shape of the boundary which poses the greatest difficulty in proving the result.[14]

I solve the problem using a novel approach, which proceeds in three steps. By Lemma 1, the pair $(z, U_A)$ is a sufficient statistic for total welfare. In the first step, I therefore characterize the feasible $(z, U_A)$ pairs. Then, in the second step, I fix a boundary $z$ and find the welfare-maximizing $U_A$ that is compatible with it. Finally, I optimize over the space of feasible pairs $(z, U_A)$, where $U_A$ is the optimal $A$-indirect utility associated with $z$. I use optimal control tools to show that the optimal boundary $z^*$ has to be linear with a slope of 1, which in turn is optimally implemented without damages.

**Characterizing feasible $(z, U_A)$.** We say $(z, U_A)$ is *feasible* if there exists a mechanism $(c, x, y)$ with $A, B$-indirect utilities $U_A, U_B$ such that:

$$U_A(a) = U_B(z(a)) \quad \text{for all} \quad a \in [\underline{a}, \overline{a}]. \tag{5}$$

**Lemma 2.** *The pair $(z, U_A)$ is feasible if and only if:*

1. *$U'_A$ and $U'_A/z'$ are non-decreasing,*
2. *The boundary $z$ has finite, strictly positive one-sided derivatives at every $a \in (\underline{a}, \overline{a})$, and a finite, strictly positive left derivative at $\overline{a}$.*
3. *The supply constraint (S′) holds:*

$$\int_{\underline{a}}^{1} \int_{0}^{\hat{z}(a)} f(a, v) dv \, da \leq s_A, \quad \int_{\underline{b}}^{1} \int_{0}^{\hat{z}^{-1}(b)} f(v, b) dv \, db \leq s_B. \tag{S′}$$

[14]The difference between my approach and the Myersonian one is suggested by Assumption 1, which is imposed on the (conditional) *anti*-hazard rate, and not on the hazard rate, as it would be in a standard Myersonian setting with a welfarist objective. The reason for why the anti-hazard rate appears in my condition, explained in the course of the proof, is logically distinct from the one for the presence of the hazard rate in the Myersonian problem.

Let us first discuss 1. Note (5) means that $U_A$ and the boundary $z$ uniquely pin down the $B$-indirect utility $U_B$. Moreover, differentiating (5) gives:

$$U_A'(a) = U_B'(z(a)) \cdot z'(a) \quad \Rightarrow \quad U_B'(z(a)) = \frac{U_A'(a)}{z'(a)}. \tag{6}$$

We know, however, that $U_B$ must be convex and, since $z$ is increasing, this implies $U_A'/z'$ has to be increasing. Thus, 1. boils down to a monotonicity condition on $U_A'$ and $U_B'$. 2. then uses (6) to establish that any boundary implemented by $U_A$ and $U_B$ has to have certain regularity properties. Finally, 3. changes the way we express supply constraints. Rather than look at good allocations $y(a,b)$ directly, it takes advantage of the fact that types who get good $A$ ($B$) are below (above) the boundary. It then measures the masses of agents getting either good by integrating over agents below and above $z$ (Figure 5).

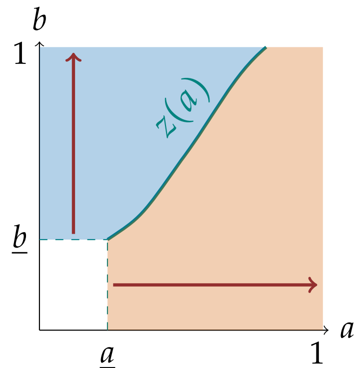


Figure 5: The supply condition (S′) ensures that the probability masses below the boundary (orange) and above it (blue) are at most $s_A$ and $s_B$, respectively. The red arrows mark the directions of integration in the left-hand-sides in (S′).

**Role of piecewise differentiability.** The results shown so far did not rely on Assumption 2, which restricts the designer to piecewise continuously differentiable allocation rules $x(a,b)$.[15] It is, however, necessary for the results that follow. The role of this assumption is to guarantee the following regularity condition on the boundary $z$:

**Fact 1.** *Under Assumption 2, every implementable boundary is piecewise twice continuously differentiable with strictly positive one-sided derivatives on* $(\underline{a}, \overline{a}]$.

This, in turn, lets us optimize over the space of boundaries using optimal control techniques.

[15]In the appendix, I provide a version of Lemma 2 under this additional restriction.

**Optimal $U_A$ for a fixed boundary $z$.** Let us now fix a boundary $z$ and find the $A$-indirect utility $U_A$ that maximizes total welfare (3) subject to $(z, U_A)$ being feasible.

**Lemma 3.** *Fix a piecewise twice continuously differentiable boundary $z$. Then $(z, U_A)$, with $U_A$ defined by (7), maximizes total welfare (3) among all feasible pairs $(z, \check{U}_A)$.*

$$U_A'(a) = \begin{cases} 0, & a \in (0, \underline{a}), \\ m(a) \cdot k, & a \in (\underline{a}, \overline{a}), \\ 1, & a \in (\overline{a}, 1). \end{cases} \tag{7}$$

*where*

$$m(a) = \exp\left( \int_{\underline{a}}^{a} \max\left[0, \frac{z''(s)}{z'(s)}\right] ds \right) \prod_{\substack{z'^{+}(t) > z'^{-}(t), \\ t \le a}} \frac{z'^{+}(t)}{z'^{-}(t)}, \qquad k = \frac{1}{\max[m(\overline{a}),\ m(\overline{a})/z'^{-}(\overline{a})]}.$$

I now explain the intuition behind this lemma. First, note that $U_A'$ is constant on all intervals where $z$ is concave and is proportional to $z'$ on all intervals where $z$ is convex. Recall also that

$$U_B'(z(a)) = \frac{U_A'(a)}{z'(a)}. \tag{8}$$

Thus, we can equivalently say that $U_A'(a)$ is constant on concave intervals of $z$ while $U_B'(z(a))$ is constant on its convex intervals (Figure 6).

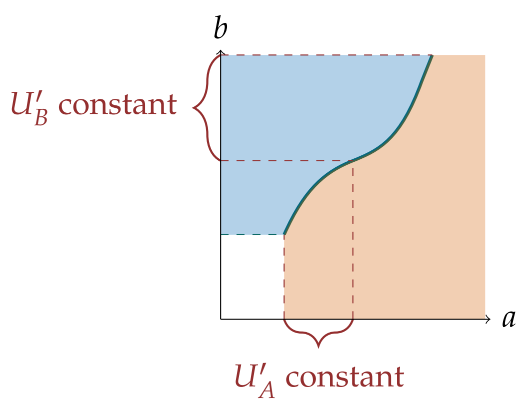


Figure 6: $U_A'(a)$ and $U_B'(z(a))$ are constant where the boundary $z$ is concave and convex, respectively.

Now, recall that by Lemma 2, $U_A'(a)$ and $U_B'(z(a))$ must be increasing. The above observation therefore means that at least one of these monotonicity constraints must always bind. Intuitively, were neither constraint to bind on some interval, we could increase $U_A'(a)$ and $U_B'(z(a))$

pointwise in a (8)-preserving manner until one of them started binding. This would create pointwise higher utility profiles and thus produce superior welfare (3).

To understand the economic intuition behind this logic, note that by the envelope theorem, $U_A'(a)$ corresponds to the quality $x$ of the goods received by agents who get good $A$ and value it at $a$. Consider then some type $(a', b')$ on the boundary and suppose that the boundary is convex on some interval $[a'', a']$. Since $U_B'(z(a))$ is always non-decreasing, (8) implies that $U_A'(a)$ has to be strictly increasing on this interval below $a'$, and so that agents with $a < a'$ get a damaged version of good $A$. This means that, in general, making a boundary curve up or down somewhere requires damaging the good on at least one side of the boundary there. Note, however, that altering the shape of the boundary could, in principle, be beneficial despite this damage, as it might entice agents to choose between the $A$- and $B$-goods in a more socially efficient way. Nevertheless, more damage cannot be better *conditional on implementing the same boundary*. Consequently, a fixed boundary is implemented most efficiently by a mechanism that damages goods as little as possible while still satisfying (8) and both monotonicity constraints.

**Showing the optimal boundary is linear.** Having pinned down the optimal $U_A$ for a given $z$, we can turn to searching for the best boundary with its optimal $A$-indirect utility. As I explain shortly, the following result relies on the anti-hazard-rate condition imposed by Assumption 1.

**Proposition 2.** *The optimal boundary* $z^* : [\underline{a}^*, \overline{a}^*] \to [\underline{b}^*, \overline{b}^*]$ *is linear.*

In what follows, I present the core idea behind the proof of this proposition. First, however, I explain the economic effects of deforming a linear boundary. To that end, consider a linear boundary with a slope $s > 1$ (Figure 7a) and introduce a kink at $\hat{a}$ bending it downwards to the slope $s' \in (1, s)$ (Figure 7b).

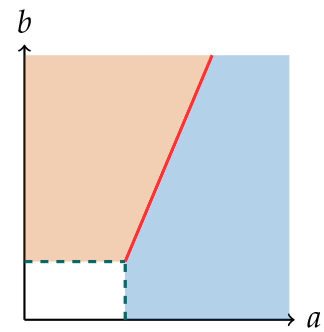


Figure 7a

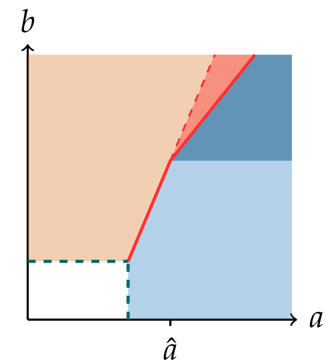


Figure 7b

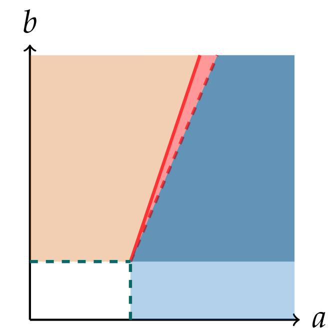


Figure 7c

Lemma 3 pins down the optimal way of implementing these boundaries. The linear boundary on the left panel is optimally implemented by offering good $A$ undamaged, while damaging good $B$ to quality $1/s < 1$ (where both options come with some ordeals). The boundary in the central panel is optimally implemented by expanding this menu by an additional option: good

$B$ with higher quality $1/s'$ and a higher ordeal. What are the effects of introducing this third option? First, some agents who were not willing to take good $B$ at quality $1/s$ are attracted by the $B$-option with the higher quality of $1/s'$. Thus, the agents in the red-shaded area in Figure 7b switch from getting good $A$ to getting good $B$. This increases their welfare, as well as tightens the supply constraint on good $B$ while relaxing it for good $A$. Second, the inframarginal takers of good $B$ whose values for it are high can now get a quality upgrade, which increases their utilities. These types are shaded dark blue in Figure 7b.

When deciding whether to introduce such a kink, the designer must therefore balance the welfare gains for the agents in the red and dark blue areas against the effect on supply constraints from the "switchers" in the red region. In particular, for the kink to be beneficial, the designer has to prefer it to a different change: tilting the whole linear boundary (Figure 7a). Such a perturbation could be implemented by increasing the quality of good $B$ in the two-option menu and adjusting the ordeals. Which of these options is better depends on the shape of the distributions of values. In particular, the kink perturbation would be preferred if agents in dark blue area in Figure 7b had sufficiently high average values for good $B$, relative to those of agents in the dark blue area in Figure 7c. However, the anti-hazard-rate conditions imposed by Assumption 1 ensure this is not the case.

More generally, Assumption 1 guarantees that introducing *any* bend to a linear boundary is always dominated by switching to a different linear one. I prove this by considering optimal control problems of optimizing over the shape of the boundary in regions where it is concave/convex. I show that no boundary with strictly convex/concave parts or kinks can satisfy the necessary optimality conditions, and thus that the optimal boundary has to be linear.

While the formal argument is relegated to the appendix, I provide an informal sketch capturing the core intuition behind it and explain why the anti-hazard-rate condition appears. Consider some interval $[\underline{a}, \overline{a}]$ on which the boundary is concave and, for simplicity, assume it consists of multiple small, linear pieces (Figure 8a). We will consider the perturbations to these linear pieces on this part of the boundary. Notice, however, that such perturbations have to respect the supply constraint (S′), and thus must preserve the probability mass below and above the boundary. Still, we can construct perturbations preserving the supply constraint by perturbing one piece of the boundary upwards and another one downwards in a ratio that leaves the probability masses unchanged (Figure 8b). First-order optimality conditions then tell us that, when perturbing one such piece, we can capture the effect of this perturbation on the supply constraint by a Lagrange multiplier $\mu$.

Now, recall that by Lemma 1, total welfare is given by (3):

$$U_A(1) - \int_0^1 U_A'(a) \cdot F(a, \hat{z}(a)) da.$$

Moreover, Lemma 3 tells us that $U_A'$ is equal to some constant on the region where the boundary is concave. Thus, we can write the effect of the boundary in the region $[\tilde{a}, \tilde{a} + \delta]$, up to scaling,

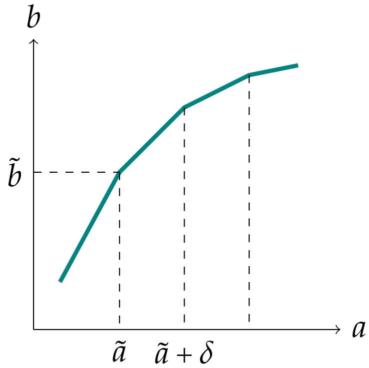

Figure 8a

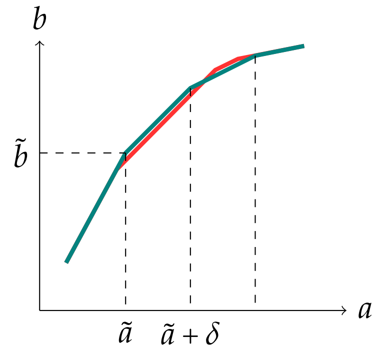

Figure 8b

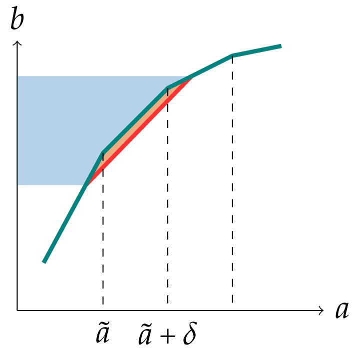

Figure 8c

as follows:

$$-\int_{\tilde{a}}^{\tilde{a}+\delta} F\left(a, \tilde{b} + (a - \tilde{a}) \cdot s\right) da.$$

Now, consider the effect of a small downward perturbation to the height of the boundary on this interval, as illustrated in Figure 8c. The first-order effect of this perturbation is given by:

$$\frac{d}{d\tilde{b}} \int_{\tilde{a}}^{\tilde{a}+\delta} F\left(a, \tilde{b} + (a - \tilde{a}) \cdot s\right) da \; - \; \mu \cdot \frac{d}{d\tilde{b}} \int_{\tilde{a}}^{\tilde{a}+\delta} \int_{0}^{\tilde{b}+(a-\tilde{a})\cdot s} f(a,b) db da.$$

The latter term captures the effect of the perturbation on the probability mass under the boundary, which is valued according to the aforementioned multiplier $\mu$ on the supply constraint (S). Performing a small perturbation like this (in either direction) is not beneficial when:

$$0 \;\approx\; \int_{\mathcal{K}} f(a,b) \, \mathrm{d}(a,b) \;-\; \mu \int_{\mathcal{L}} f(a,b) \, \mathrm{d}(a,b)$$

where $\mathcal{K}$ and $\mathcal{L}$ are, respectively, the blue region in Figure 8c and the orange region in Figure 8c. Dividing through by the latter integral then reduces the condition to:

$$0 \;=\; \frac{\int_{\mathcal{K}} f(a,b) \, \mathrm{d}(a,b)}{\int_{\mathcal{L}} f(a,b) \, \mathrm{d}(a,b)} \;-\; \mu.$$

Now, notice that when the length $\delta$ of the perturbed interval becomes small, we can use the following approximation:

$$\frac{\int_{\mathcal{K}} f(a,b) \, \mathrm{d}(a,b)}{\int_{\mathcal{L}} f(a,b) \, \mathrm{d}(a,b)} \;\approx\; \frac{\int_0^{\tilde{a}} f(v, z(\tilde{a})) dv}{f(\tilde{a}, z(\tilde{a}))} \;=\; \frac{F_{A|B}(\tilde{a}|z(\tilde{a}))}{f_{A|B}(\tilde{a}|z(\tilde{a}))}.$$

Thus, when the boundary is strictly concave on some region, a profitable perturbation analogous to that in Figure 8b does not exist only if at every point in that interval we have:

$$0 = \frac{F_{A|B}(\tilde{a}|z(\tilde{a}))}{f_{A|B}(\tilde{a}|z(\tilde{a}))} - \mu.$$

However, this cannot be the case by Assumption 1. Recall the inverse anti-hazard rate is strictly increasing in one of the variables, non-decreasing in the other one, and $z$ is strictly increasing in $a$. Thus, if we were indifferent about perturbing the boundary slightly up or down at some level of $a$, we would strictly prefer to perturb it upwards for any higher $a$.

The proof formalizes this reasoning in the continuous case using optimal control methods. The intuition presented here explains why the boundary does not have strictly convex and concave intervals. The argument extends this reasoning to show the boundary cannot have kinks.[16]

**The optimal linear boundary has a unit slope.** Lemma 3 pins down the $U'_A$ optimally implementing any linear boundary. In particular, it shows that when the slope $s$ of the linear boundary is at least 1, good $A$ will not be damaged and good $B$ will be given out with quality $1/s$. It therefore remains to find the optimal slope of the linear boundary.

**Lemma 4.** *The optimal boundary $z^*$ is linear with a slope of* 1*.*

I now give an intuition for this result. Suppose the boundary $z$ was initially sloped at $s > 1$ and reached the ceiling of the unit square before it reached its right wall (Figure 9a).

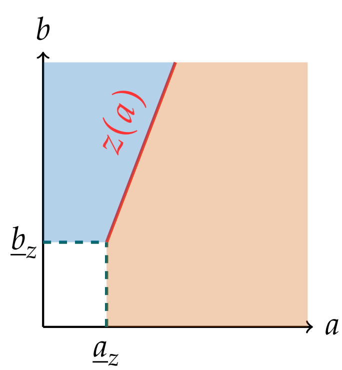


Figure 9a: $s$-sloped boundary $z$.

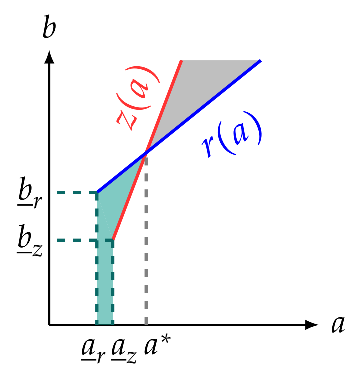


Figure 9b: Regions $\overline{\mathcal{D}}$ (gray) and $\underline{\mathcal{D}}$ (green).

A mechanism with such a boundary can be implemented by offering two options: good $A$ with no damage and an ordeal of $\underline{a}_z$, and good $B$ with quality $1/s$ at an ordeal of $\underline{b}_z/s$. I now explain why such a boundary with a slope exceeding 1 cannot be optimal under Assumption 1. To that end, consider a slightly less steep boundary $r$ such that the same amount of each good is

[16] Note that the presented intuition never required the inverse anti-hazard rate be strictly *increasing*, but just that it be strictly *monotonic*. In fact, the argument for why strictly convex and concave intervals cannot be optimal survives in the case where inverse anti-hazard rate is strictly decreasing. The latter part of the argument, which shows why the boundary cannot be piecewise linear with kinks, requires it to be increasing.

allocated under it as it was under $z$ (Figure 9b). A few features of this boundary are apparent. First, it crosses $z$ once, and from above; let $a^*$ denote this crossing point. Second, its lowest participating value for $A$ is lower: $\underline{a}_r < \underline{a}_z$. Finally, consider the green and gray regions in Figure 9b; I refer to them as $\underline{\mathcal{D}}, \overline{\mathcal{D}}$:

$$\underline{\mathcal{D}} = \left\{(a,b): 0 < a < a^*,\ \hat{z}(a) < b < \hat{r}(a)\right\} \quad \overline{\mathcal{D}} = \left\{(a,b): a^* < a < 1,\ \hat{r}(a) < b < \hat{z}(a)\right\}.$$

These denote the types who received good $B$ under $z$ and receive good $A$ under $r$, and vice versa. Since the masses of agents who receive each good are equal for both boundaries, the masses of agents in these two regions must be equal as well:

$$\int_{\overline{\mathcal{D}}} f(a,b)\,\mathrm{d}(a,b) \;=\; \int_{\underline{\mathcal{D}}} f(a,b)\,\mathrm{d}(a,b). \tag{9}$$

Now, let us consider the difference $\Delta$ between the value of the objective (3) for boundaries $r$ and $z$. As explained previously, both mechanisms' $U_A'$ correspond to the qualities of good $A$, and thus is uniformly equal to 1 as good $A$ is undamaged in both mechanisms. The difference can then be written as:

$$\begin{aligned}
\Delta &= (1-\underline{a}_r) - \int_0^1 F\left(a,\hat{r}(a)\right)\,da - \left((1-\underline{a}_z) - \int_0^1 F\left(a,\hat{z}(a)\right)\,da\right)\\
&= (\underline{a}_z - \underline{a}_r) - \left(\int_0^1 F\left(a,\hat{r}(a)\right) - F\left(a,\hat{z}(a)\right)\,da\right)\\
&= (\underline{a}_z - \underline{a}_r) - \left(\int_{\underline{\mathcal{D}}} \frac{F_{A|B}(a\mid b)}{f_{A|B}(a\mid b)}\cdot f(a,b)\,\mathrm{d}(a,b) \;-\; \int_{\overline{\mathcal{D}}} \frac{F_{A|B}(a\mid b)}{f_{A|B}(a\mid b)}\cdot f(a,b)\,\mathrm{d}(a,b)\right).
\end{aligned}$$

We can therefore decompose the difference in welfare between the boundaries into two effects: $\underline{a}_z - \underline{a}_r$, which corresponds to the change in good $A$'s ordeal, and the difference in integrals which corresponds to the change in the total values of the goods to their recipients, taking into account the change in their qualities and the change in recipient composition.

Assumption 1 lets us sign these two effects separately. Indeed, as mentioned, $\underline{a}_r < \underline{a}_z$, so the former effect is positive. To see how to sign the latter effect, recall that Assumption 1 states that

$$\frac{F_{A|B}(a\mid b)}{f_{A|B}(a\mid b)}, \tag{10}$$

is weakly increasing in both $a$ and $b$. Note also that the region $\overline{\mathcal{D}}$ lies to the north-east of $\underline{\mathcal{D}}$, and thus the values of (10) are uniformly higher in it. Moreover, $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ contain equal probability masses by (9), and so it follows that the latter integral must be higher.

For this reason, the optimal boundary has a slope of 1. Lemma 3 then tells us the mechanism implementing it does not use damages.

**Discarding goods is never optimal.** Finally, we need to show that the optimal mechanism allocates the whole supply of both goods. As we have already shown, this mechanism uses only ordeals to enforce the supply constraints; we can interpret these ordeals as prices. Then, intuitively, if the designer were to discard some of one good's supply, she could do better by simply lowering its associated ordeal and letting demand for it increase. As a result the designer selects the (unique) pair of ordeals at which the markets for both goods clear.

**Lemma 5.** *The optimal mechanism allocates the whole supply of both goods.*

## 5 Damages can be optimal

Theorem 1 rules out damages under specific distributional assumptions. The main result below shows that if these assumptions fail, damaging goods may be optimal.

**Theorem 2.** *There exist supplies $s_A, s_B > 0$ and a type distribution $F$ with full support on $[0,1]^2$ for which the optimal mechanism features damages, that is, $x(a,b) < 1$ for a positive mass of agents.*

The proof, presented in the appendix, is based on the following example density:

$$f(a,b) = \begin{cases} \epsilon \dfrac{2}{\left(\frac{1}{2}-\epsilon\right)^2}, & \text{if } b-a \geq \frac{1}{2}+\epsilon, \\ \zeta \dfrac{2}{\epsilon-\epsilon^2}, & \text{if } b-a \in \left[\frac{1}{2}, \frac{1}{2}+\epsilon\right), \\ \frac{8}{7}(1-\zeta-\epsilon), & \text{if } b-a < \frac{1}{2}, \end{cases} \tag{11}$$

with $\zeta \in (0, 1-\epsilon)$ and supplies given by $s_A = 1-\zeta-\epsilon$ and $s_B = \zeta+\epsilon$.

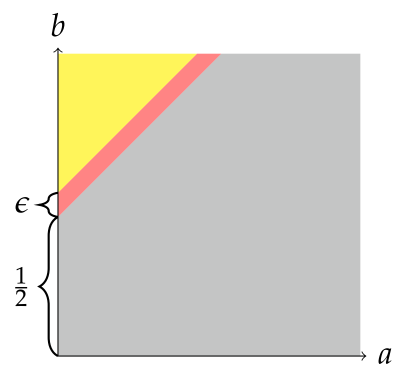


Figure 10: Distribution $F$ from (11).

This distribution is illustrated in Figure 10. The probability masses in all three shaded areas are constant in $\epsilon$: they are equal to $\epsilon$ in the yellow area, $\zeta$ in the red area, and $1-\zeta-\epsilon$ in the gray one. The supply of good $B$ is chosen to exactly match the total mass of the yellow and red areas, while the supply of good $A$ matches the mass in the gray area.

Consider a mechanism for this distribution which uses only ordeals but not damages. Discarding any of either good's supply would not be helpful (this point is later shown formally in the proof of Theorem 1), so we can without loss consider only mechanisms where the whole supply is allocated. Without damages, this can only be achieved by giving out good $A$ for free, and giving good $B$ with an ordeal $c = 1/2$. Indeed, this mechanism induces a pattern of sorting illustrated in Figure 11a, with types shaded in orange getting good $B$ and types shaded in blue getting good $A$. As discussed above, the boundary splitting the two regions is angled at 45 degrees.

Now, notice that for agents in the strip between the solid and dashed lines in Figure 11a, the surplus from getting good $B$ over getting good $A$ is at most $\epsilon$. This is because most of their surplus is consumed by the ordeal $c = 1/2$. Consider then the case where $\epsilon \to 0$. As $\epsilon$ falls, this surplus goes to zero for the whole mass $\zeta$ of agents in the aforementioned strip. Similarly, the mass of agents above the dashed line, equal to $\epsilon$, also tends to zero. Consequently, total welfare then tends to that which would have resulted from all agents getting good $A$ for free.

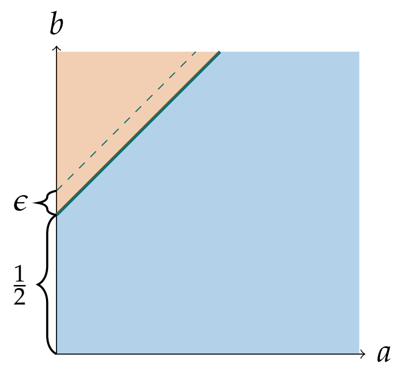


Figure 11a: The mechanism for distribution (11) that gives good $B$ with an ordeal.

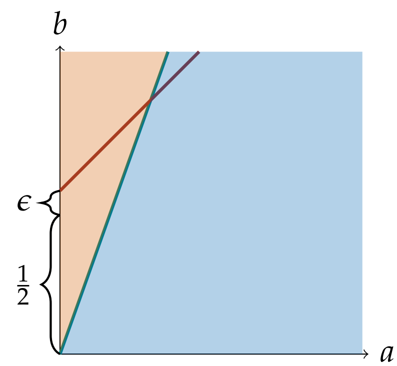


Figure 11b: The mechanism for distribution (11) that damages good $B$.

Consider by contrast the mechanism which uses no ordeals but damages good $B$ to the point where the resulting boundary satisfies both goods' supply constraints (Figure 11b). Here too, total welfare of agents above the red line becomes negligible as $\epsilon \to 0$. Note, however, that agents who are below the red line and far to the left of the green line now benefit substantially from getting good $B$ over getting good $A$ for free. These agents have a strong *relative* preference for $B$ over $A$, and thus strongly prefer even damaged $B$ to undamaged $A$. Since the mass of such agents does not go to zero as $\epsilon$ decreases, this mechanism creates substantial gains from allocating good $B$ over $A$ even in the limit case. This is in contrast to the ordeal-only mechanism, where all the gains from allocating $B$ over $A$ are "eaten away" by the ordeal used to screen high-$b$ types, thus making the damage-based mechanism superior. Intuitively, this occurs because most of the consumers of good $B$ under the ordeal-based mechanism have values very close to the "market-clearing ordeal".

Finally, to see why the distribution in (11) violates Assumption 1 necessary for Theorem 1, recall that this assumption rules out sharp spikes in the conditional density $f(b|a)$ as $b$ increases. Our density, however, has exactly such a spike when $b$ crosses the line defined by $b - a = 1/2$, for $a$ small enough.

## 6 One-dimensional heterogeneity

Let us now consider the case where only good $A$ is scarce, and interpret good $B$ as an unlimited outside option for which all agents have common values. I therefore assume $s_A < 1$ and $s_B = \infty$, and set $b > 0$ to be all agents' value for good $B$.

**Proposition 3.** *Any mechanism that uses damages, so features $x(a,b) < 1$, is Pareto-dominated by a mechanism that uses only ordeals, i.e. where $x(a,b) \equiv 1$.*

*Proof.* By single crossing, there is a cutoff $\underline{a}$ such that all and only agents with value $a$ above it get good $A$:

$$\underline{a} := \inf\{a \in [0,1] : y(a,b) = A\}, \quad \text{with } \inf\varnothing := 1.$$

Fix the value for the cutoff $\underline{a}$; I show that among mechanisms with this cutoff, the one not using damages Pareto-dominates all the other ones. First, note that for the mechanism to satisfy (IC), all agents who do not receive $A$ must be getting the same utility level $\underline{U}$. By the envelope theorem, we can then write an agent's indirect utility as:

$$U(a,b) = \underline{U} + \int_{\underline{a}}^{\max[\underline{a},a]} x(v,b)\, dv.$$

Now, notice that $x(a,b) \leq 1$, and thus $U(a,b)$ is at most $\underline{U} + \max[\underline{a},a] - \underline{a}$. Indeed, this upper bound on the utility profile is implementable by offering a menu with two options: good $B$ with $x = 1, c = b - \underline{U}$ and good $A$ with $x = 1$ and $c = \underline{a} - \underline{U}$. □

To understand the intuition behind this result, note that in the single-good case, good $A$ will go to agents whose values $a$ for it lie in some upper interval $[\underline{a}, 1]$ under any mechanism. The designer's problem therefore boils down to selecting a cutoff $\underline{a}$ and choosing how to enforce it. To do so, she has to deter some low-$a$ agents from choosing good $A$. The burden imposed on the takers of $A$ must then be chosen to make the cutoff type $\underline{a}$ exactly indifferent between $A$ and the outside option. This can be done by imposing an ordeal on those choosing good $A$, or by damaging good $A$ to make the outside option relatively more attractive. Note, however, that damages are more burdensome to *inframarginal types* than the types below $\underline{a}$ they are actually meant to deter. Ordeals, on the other hand, are equally burdensome to everyone. Thus, enforcing the cutoff $\underline{a}$ with ordeals leaves more rents to inframarginal takers of $A$ (Figure 12).[17]

[17] Proposition 3 was formulated in a linear model to maintain consistency with the two-good setup. However,

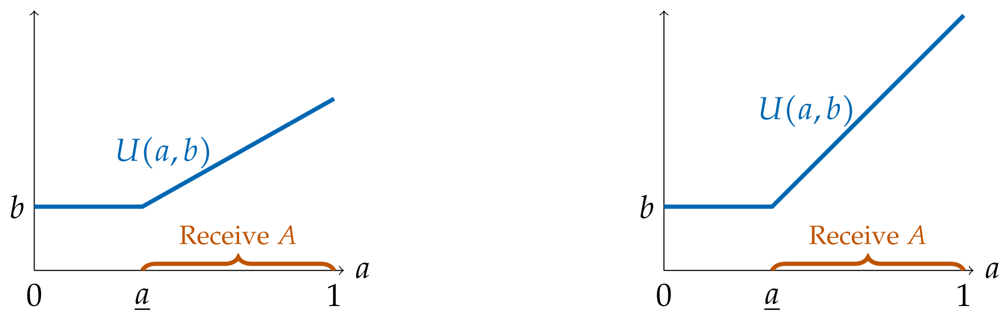


Figure 12: Indirect utilities $U(a,b) = b + \int_{\underline{a}}^{\max[\underline{a},a]} x(v,b)\, dv$ for mechanisms enforcing the cutoff $\underline{a}$ with damages (left) and ordeals (right).

While Proposition 3 is simple, it provides economic insight. For example, it captures a key force in the model of congestion pricing of Vickrey (1973) and its subsequent generalization by Van Den Berg and Verhoef (2011). The model studies the use of congestion pricing to spread the amount of traffic passing through a capacity-constrained road. It shows that when drivers' values for time are identical, congestion pricing does not improve their utility. This is because the same allocation of driving times is implemented with or without tolls: the "prices" for traveling at specific times are then pinned down by market clearing, and it is irrelevant whether they are paid in money, through tolls, or in wasted time, through congestion. This conclusion is overturned, however, when agents' values for time are heterogeneous. Proposition 3 makes this clear: while tolls are an *ordeal*, waiting in traffic is a *damage*. When the road's capacity constraint at peak time is enforced through payments, everyone pays the same price as the marginal driver. When it is enforced through waiting in traffic, the marginal driver experiences the same disutility as she would from the payment, but the inframarginal drivers with the highest values for arriving early suffer strictly more.

### 6.1 Contrast with two-dimensional heterogeneity

Proposition 3 contrasts with the results of the previous sections: when agents differ only in their values for one good, damages are never optimal. When they have heterogeneous values for both of them, the designer can sometimes benefit from damaging goods.

I now explain the intuition behind this difference. First, note that in the case of a single scarce good, all implementable patterns of "sorting" into good $A$ and the outside option $B$ can be implemented using both ordeals and damages. Indeed, the proof of Proposition 3 exploits this fact by first pinning down a sorting pattern and then showing that a mechanism using only ordeals implements it optimally. This equivalence, however, is specific to the one-dimensional

its logic easily extends to the case of general screening instruments whose disutilities to agents are additive. We could then consider two wasteful screening instruments, where the cost of one is increasing more steeply in the value for $a$ than the cost of the other. The result would then say that any mechanism using the former instrument is Pareto-dominated by one using only the latter instrument.

case. With two-dimensional heterogeneity, there is in general no uniform ranking of agents that determines the sorting into the available goods. Rather, damages and ordeals make agents sort into options in distinct ways. By combining both instruments, the designer can then implement substantially richer sorting patterns than with ordeals alone.

To see that damages and ordeals sort agents differently, consider first the case in which the designer does not use damages, i.e. $x(a,b) \equiv 1$. In this case, (IC) requires that each good be given with a single ordeal, $c_A$ or $c_B$. Agents of type $(a,b)$ will then select good $A$ if:

$$a - c_A > b - c_B,$$

and select good $B$ otherwise. Note that this kind of screening can only lead to sorting patterns like the one illustrated in Figure 13a, where agents get good $A$ if their types lie below a certain 45-degree line. It cannot create a sorting pattern like that in Figure 13b, where agents get good $A$ if their types lie below a ray from the origin, that is, if:

$$\frac{b}{a} < q,$$

for some $q$, and get good $B$ otherwise. Such a sorting pattern can be achieved with damages, however. Consider a mechanism offering $A$ with a damage, $x = q < 1$, and $B$ without it, $x = 1$, with no ordeals for either. The set of indifferent agents will then be given by:

$$a \cdot q = b \quad \Rightarrow \quad \frac{b}{a} = q.$$

Agents below and above this boundary will then choose goods $A$ and $B$, respectively.

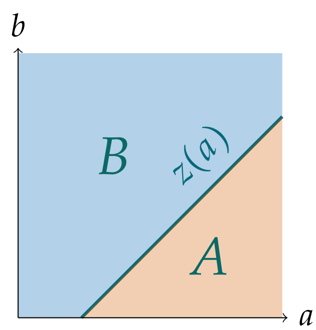

Figure 13a: Sorting pattern implementable with ordeals alone.

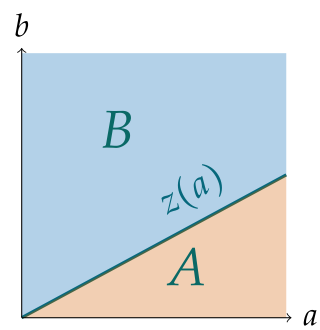

Figure 13b: Sorting pattern whose implementation requires damages.

The fact that achieving certain sorting patterns requires using damages breaks the intuition underlying Proposition 3. Indeed, as shown in Theorem 2, sorting patterns that are only implementable using damages can sometimes prove optimal.

# 7 Extensions

I now show that the main insights of my model continue to hold once additional economic forces are introduced. In particular, I consider the case where ordeals are not fully wasteful, allow for heterogeneous ordeal costs, and introduce savings from quality reduction.

## 7.1 Partially wasteful ordeals

I now consider the case where every unit of the ordeal generates a benefit $\gamma \in [0,1]$ for the designer; I interpret this partially wasteful screening device as a monetary payment and let $\gamma$ be the designer's value for the collected revenue. If this revenue could be costlessly rebated to participants, it would be natural to set $\gamma = 1$. This would in turn mean that the transfers in the mechanism are welfare-neutral, effectively reducing the designer's objective to maximizing allocative efficiency. In this case, we recover the standard result about the optimality of market mechanisms. Since the allocation of the mechanism in Theorem 1 replicated the competitive equilibrium allocation, we get the following result:

**Proposition 4.** *The mechanism described in Theorem 1 maximizes allocative efficiency* (E)*:*

$$\int \mathbb{1}_{y(a,b)=A} \cdot x(a,b) \cdot a + \mathbb{1}_{y(a,b)=B} \cdot x(a,b) \cdot b \;\; dF(a,b), \tag{E}$$

*subject to the supply constraint* (S)*.*

In practice, however, we might expect the designer to consider transfers wasteful. For government programs, this could be the case when rebating revenue to participants is possible but costly due to bureaucratic inefficiency, or because distributing cash lacks the screening benefits of in-kind transfers.[18] In such cases, a redistributive designer might not want to use the program to collect revenue, as its participants are significantly poorer than the average taxpayer.[19] These concerns can be modelled by considering payments a *partially* wasteful instrument, with a weight $\gamma \in [0,1]$ on revenue in the designer's objective:

$$\int u_{a,b}(a,b)\; dF(a,b) \; + \; \gamma \cdot \int c(a,b)\; dF(a,b). \tag{12}$$

[18] When the designer hands out a free (or subsidized) inferior good, only relatively poor agents will want to participate as wealthier ones can afford higher-quality alternatives. Thus, the subsidy is automatically targeted to those who need it most (Besley and Coate, 1991). As soon as the designer hands out cash, such targeting disappears as money is desired by everyone, regardless of wealth.

[19] While the model does not explicitly account for wealth differences or heterogeneous welfare weights among agents, this can be viewed as an approximation of a scenario where such differences exist but are relatively small *between* participants compared to the gap between participants and the average taxpayer. This is especially likely when the designer allocates inferior goods, such as public housing in undesirable areas. The designer's welfare-weighted objective can then be approximated by a constant weight on all participants, which is distinct from that on revenue, representing her welfare weight for the average taxpayer.

As may be intuitive, including partial revenue considerations in the designer's objective does not break Theorem 1:

**Corollary 1.** *Theorem 1 continues to hold when the designer maximizes (12) for any $\gamma \in [0,1]$.*

The corollary follows because (12) is a linear combination of the objectives (W) and (E), each of which is maximized by the ordeal-only mechanism (by Theorem 1 and Proposition 4, respectively). The same mechanism must therefore also maximize their combination.

### 7.2 Heterogeneous ordeal costs

Certain ordeals are more costly to some agents than to others. For instance, poorer agents who benefit most from social programs may find monetary payments more burdensome. I extend the model to accommodate such heterogeneous ordeal costs: suppose agents' types are three-dimensional, $(a,b,r)$, where $r$ represents an agent's unit cost for the ordeal. Utilities are then given by:

$$x \cdot a - r \cdot c \;\text{ if }\; y = A,$$
$$x \cdot b - r \cdot c \;\text{ if }\; y = B.$$

I assume $r$ is distributed on $[\underline{r}, \bar{r}]$ with $\underline{r} > 0$. Types $(a,b,r)$ are then distributed according to $\tilde{F}$ with full support on $[0,1]^2 \times [\underline{r}, \bar{r}]$.

Fortunately, the methods from the baseline model carry over to this enriched setup. Following Dworczak et al. (2021), observe that all agents with types $(a,b,r)$ and $(a/r, b/r, 1)$ are behaviorally equivalent. I can then adopt an approach similar to theirs: I renormalize all agents' ordeal cost to one, and identify agents with transformed values

$$(\tilde{a}, \tilde{b}) = \left(\frac{a}{r}, \frac{b}{r}\right),$$

and a welfare weight $\lambda(\tilde{a}, \tilde{b})$ equal to the expectation of $r$ among agents whose types were renormalized to $(\tilde{a}, \tilde{b})$:

$$\lambda(\tilde{a}, \tilde{b}) = \mathbb{E}\left[ r \;\middle|\; \frac{a}{r} = \tilde{a},\; \frac{b}{r} = \tilde{b} \right].$$

Denote by $G$ the distribution of transformed types implied by $\tilde{F}$; note that $G$ has full support on $[0, 1/\underline{r}]^2$. The designer's objective then reduces to maximizing total *weighted* welfare:

$$\int \lambda(\tilde{a}, \tilde{b})\; u_{\tilde{a},\tilde{b}}(\tilde{a}, \tilde{b})\; dG(\tilde{a}, \tilde{b}). \tag{13}$$

We then get the following result, which is a direct analog of Theorem 1:

**Corollary 2.** *Suppose Assumption 2 holds and that there are fewer goods than agents: $s_A + s_B \le 1$. Moreover, assume that $G$ has a Lipschitz continuous density $g$, the ratios*

$$\frac{\int_0^a \lambda(v,b)\cdot g(v,b)dv}{g(a,b)}, \quad \frac{\int_0^b \lambda(a,v)\cdot g(a,v)dv}{g(a,b)},$$

*are increasing in $a$ and $b$, and for each ratio at least one of the two monotonicities is strict. Then the optimal mechanism offers only two options: it allocates undamaged goods $A$ and $B$ with ordeals $c_A$ and $c_B$. These ordeals are chosen so that the whole supply of both goods is allocated.*

The conditions of the corollary are analogous to those of Theorem 1, except the density of types is multiplied by the welfare weights the planner attributes to them. While the proof is almost entirely analogous, I outline the key modifications in the appendix.

### 7.3 Savings from quality reduction

The baseline model assumed that reducing quality does not generate monetary savings for the designer. While this may be a good approximation for some forms of damages discussed, I now also consider the case where keeping quality high is costly. I assume that damages generate linear monetary savings of $r > 0$ per unit of quality reduction. My result for this setting relies on the following distributional assumption:

**Assumption 3.** *Agents' values for goods $A$ and $B$ are independent, with distributions $F_A$ and $F_B$, respectively. These distributions have continuously differentiable pdfs $f_A$ and $f_B$ and are such that:*

$$\frac{F_A(a)}{f_A(a)}, \frac{F_B(b)}{f_B(b)} \text{ are strictly increasing and } \frac{f_A'(a)}{f_A(a)}, \frac{f_B'(b)}{f_B(b)} \text{ are non-decreasing.}$$

The conditions of Assumption 3 are more demanding than those necessary for previous results. However, they still hold for certain natural distributions, such as the uniform distribution.

Under Assumptions 2 and 3, the optimal mechanism may use damages, but still maintains a simple structure:

**Proposition 5.** *Suppose Assumptions 2 and 3 hold, that there are fewer goods than agents: $s_A + s_B \le 1$, and that damages generate linear monetary savings of $r > 0$ per unit of quality reduction. Then the optimal mechanism consists of at most three options: one good is always offered undamaged, while the other can be offered in at most two versions—damaged or undamaged.*

The proof of the proposition, which has been relegated to the appendix, closely follows that of Theorem 1. Intuitively, the strengthened distributional conditions let us show that the optimal boundary remains linear even when accounting for the effects of savings from quality decreases. However, they do not guarantee that the optimal linear boundary will have a slope of one. Thus, as explained when discussing the proof of Theorem 1, its implementation may require damaging one of the goods. Nevertheless, linearity of the optimal boundary guarantees that its corresponding mechanism can be implemented with a simple menu.

## 8 Conclusion

The main contribution of this paper is the distinction between two kinds of wasteful screening instruments: ordeals, which are separable from agents' values for the allocated good, and damages, which are more burdensome for agents whose values for the goods are higher. I study their screening properties when they are used together and show that screening with damages is detrimental when agents differ only with respect to the value for one good. This is no longer the case, however, when heterogeneity in values is multidimensional. I show that in such settings a novel "sorting" motivation for damaging goods emerges: the designer may reduce the quality of a certain option to ensure that only agents with a sufficiently strong relative preference for this good select it. Still, even in the multi-good case, ordeal-only mechanisms are optimal under certain regularity conditions.

These results provide suggestions for the design of social programs. First, when the allocated good is homogeneous, as is the case for e.g. food aid, energy credits, or basic medical appointments, damage instruments should be avoided in favor of ordeals. For instance, my argument suggests that usage restrictions on food stamps would be a less efficient way of screening than periodic recertification. Similarly, my analysis provides an argument against reducing the quality of government provisions or imposing usage restrictions on them when the savings from doing so would be insubstantial. For instance, many countries offer broadband subsidies restricted to low-tier plans with slow connection speeds.[20] Imposing such quality restrictions could be a less efficient means of screening than e.g. requiring more frequent renewal applications. My analysis also applies to settings with heterogeneous goods, such as affordable housing programs. It recognizes that households choosing developments based on wait-times amounts to screening with a damage instrument. While wait-times are often not an explicit design choice but equilibrium objects, they continue to exhibit screening properties described by the model. Moreover, they can be influenced by rebalancing the subsidies offered for different kinds of units. My results provide an argument for such a reform: by raising the subsidy for less-desired units and increasing the price of popular ones, the designer can bring the lengths of their waitlists closer together, and in doing so diminish the screening role of the damage instrument in the program. Instead, participants would be incentivized to choose between options based on monetary costs: a (partially) wasteful ordeal instrument.

[20]See, for instance, social tariffs in the UK or the Tarifa Social de Internet in Portugal.

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# A Waitlists

In this section, I consider a simple environment where goods and agents arrive dynamically. I consider a patient designer who optimizes steady-state welfare by selecting menus of wait-times, ordeals, and allocation probabilities for both kinds of good. I show that the designer's problem has the structure of that in the baseline model.

The designer distributes two types of goods, $A$ and $B$, with agents' values for them given by $(a,b)$. Goods and agents arrive continuously over time. At every instant $\tau \in \mathbb{R}$, flow masses $s_A, s_B > 0$ of goods $A$ and $B$ arrive, with $s_A + s_B \leq 1$. Concurrently, a unit flow mass of agents arrives, with types $(a,b)$ distributed according to a joint distribution $F$.

There are two separate waitlists, one for each good. For each waitlist, the designer chooses a menu of three objects: ordeals $c \in \mathbb{R}_+$, wait-times $t \in \mathbb{R}_+$, and the probabilities $p \in [0,1]$ with which the agent gets the good at the end of the wait. I assume that when an agent selects an option $(c,t,p)$ from a waitlist menu, she completes the ordeal immediately and then waits $t$ units of time. At the end of the wait, she receives the good with probability $p$. If she does not receive it, she may re-enter the mechanism. Like in the baseline model, type $(a,b)$ derives value $a$ from receiving good $A$ and $b$ from good $B$, and suffers additive disutilities from ordeals; agents discount their flow utilities at rate $\rho$. I assume without loss that all agents of the same type choose the same good and menu option regardless of when they arrive and how many times they have already participated.

I restrict attention to mechanisms that admit a steady state of the system. Note that such a steady state exists if and only if the mass of agents choosing each good in every period is at most the arrival rate of that good:

$$\int \mathbb{1}_{(a,b)\text{ chooses }A}\, dF(a,b) \leq s_A, \quad \int \mathbb{1}_{(a,b)\text{ chooses }B}\, dF(a,b) \leq s_B. \tag{14}$$

Crucially, the constraint is unaffected by the fact that the designer can offer goods probabilistically: since agents can always re-enter, all types selecting a menu option which gives them the good with a positive probability will get it at some point almost surely. Thus, for balance to be maintained, every agent choosing a given good must have an associated unit of that good arriving in the system in the representative period, regardless of her allocation probability.

Now, consider a type-$(a,b)$ agent choosing an option $(c,t,p)$ in waitlist $A$ (the case of $B$ is analogous). Let $V_A(a;c,t,p)$ denote her expected discounted utility at the moment she selects the option. Note that after an unsuccessful attempt the agent faces the same continuation problem (but shifted forward by $t$). Hence, $V_A$ satisfies:

$$V_A(a;c,t,p) = -c + p\,e^{-\rho t}\,a + (1-p)\,e^{-\rho t}\,V_A(a;c,t,p),$$

which gives:

$$V_A(a;c,t,p) = \frac{-c + p\,e^{-\rho t}\,a}{1-(1-p)e^{-\rho t}} = x \cdot a - \tilde{c}, \tag{15}$$

where

$$x := \frac{p\,e^{-\rho t}}{1-(1-p)e^{-\rho t}} \in [0,1], \qquad \tilde{c} := \frac{c}{1-(1-p)e^{-\rho t}} \in \mathbb{R}_+.$$

Note (15) shows that the agent's utility from an option $(c,t,p)$ depends on $(c,t,p)$ only through the pair $(\tilde{c},x)$. Since wait-times do not figure in the steady-state constraint (14), we can write the designer's problem in a direct-revelation formulation analogous to that in the main model. This reduces it to choosing steady-state allocation rules $(\tilde{c},x,y) : [0,1]^2 \to \mathbb{R}_+ \times [0,1] \times \{A,B,\varnothing\}$, subject to IC, IR and the steady-state constraint:

$$\text{for all } (a,b),(a',b') \in [0,1]^2, \quad u_{a,b}(a,b) \geq u_{a,b}(a',b'),$$

$$\text{for all } (a,b) \in [0,1]^2, \quad u_{a,b}(a,b) \geq 0,$$

$$\int \mathbb{1}_{y(a,b)=A}\, dF(a,b) \leq s_A, \quad \int \mathbb{1}_{y(a,b)=B}\, dF(a,b) \leq s_B,$$

where:

$$u_{a,b}(a',b') := \begin{cases} a\;x(a',b') - \tilde{c}(a',b') & \text{if } y(a',b') = A, \\ b\;x(a',b') - \tilde{c}(a',b') & \text{if } y(a',b') = B, \\ 0 & \text{if } y(a',b') = \varnothing. \end{cases}$$

This makes the problem identical to that in the baseline model. Note that despite allowing the designer to randomize allocations within each waitlist, the free re-entry restriction renders this tool unhelpful. This highlights that the distinction between waitlists and lotteries is contingent on the designer's ability to track agents' identities and exclude those who have already participated (see Arnosti and Shi (2020) for a related discussion).

## B Justifying the direct-revelation formulation

In the main model, I restrict the designer to deterministic menus, meaning that each menu option allocates good $A$, good $B$, or nothing with certainty. In this environment, the standard revelation principle (which assigns to each agent her favorite option from the menu) does not immediately apply, because there may exist equilibria that are more favorable for the designer in which agents randomize over menu options. Intuitively, such mixing could potentially restore some of the randomization power that the restriction to deterministic menus removes.

However, this possibility can never strictly benefit the designer. In this section, I show that for any feasible (possibly randomized) selection of agents' favorite menu options, there exists a deterministic selection that allocates the same aggregate quantities of both goods and leaves every agent's utility unchanged. Establishing that deterministic selection is without loss then allows us to apply the standard revelation argument, which in turn justifies the formulation of the designer's problem used in the main model.

Fix two closed and bounded menus $M_A, M_B \subset [0,1] \times \mathbb{R}_+$ and denote the outside option by $\varnothing$. Let $\Omega$ be the set of options available to agents:

$$\Omega := \{\varnothing\} \cup \big(\{A\} \times M_A\big) \cup \big(\{B\} \times M_B\big).$$

I write the utility of type $(a,b)$ from option $\omega \in \Omega$ as:

$$u_{a,b}(\omega) := \begin{cases} ax - c & \text{if } \omega = (A,x,c), \\ bx - c & \text{if } \omega = (B,x,c), \\ 0 & \text{if } \omega = \varnothing. \end{cases}$$

A *mechanism* is a map $\sigma : [0,1]^2 \to \Delta(\Omega)$ where $\sigma_{a,b}(\omega)$ is the probability that type $(a,b)$ selects option $\omega \in \Omega$. For any Borel set $E \subseteq \Omega$, define the induced (ex ante) mass of agents selecting an option in $E$ by:

$$D_\sigma(E) := \int \sigma_{a,b}(E)\, dF(a,b).$$

Thus, the mechanism $\sigma$ satisfies the supply constraints if:

$$D_\sigma\big(\{A\} \times M_A\big) \le s_A, \qquad D_\sigma\big(\{B\} \times M_B\big) \le s_B.$$

A mechanism $\sigma$ is *agent-optimal* if for every $(a,b)$ and $m \in \operatorname{supp} \sigma_{a,b}$ we have:

$$m \in \arg\max_{\omega \in \Omega} u_{a,b}(\omega).$$

Indeed, this maximizer set is not empty as the menus are closed and bounded. The following result shows that we can without loss restrict attention to deterministic mechanisms:

**Proposition 6.** *Fix an agent-optimal mechanism $\sigma$. Then there exists a deterministic mechanism $\tilde{\sigma}$ of the form $\tilde{\sigma}_{a,b} = \delta_{\tilde{m}(a,b)}$ for some $\tilde{m} : [0,1]^2 \to \Omega$ such that $D_{\tilde{\sigma}}\big(\{A\} \times M_A\big) = D_\sigma\big(\{A\} \times M_A\big)$, $D_{\tilde{\sigma}}\big(\{B\} \times M_B\big) = D_\sigma\big(\{B\} \times M_B\big)$, and*

$$u_{a,b}\big(\tilde{m}(a,b)\big) = \int_\Omega u_{a,b}(\omega)\, d\sigma_{a,b}(\omega) \quad \textit{for all } (a,b).$$

*Proof.* For every type $(a,b)$ define the induced interim probabilities of getting each good:

$$p_A(a,b) := \sigma_{a,b}\big(\{A\} \times M_A\big), \qquad p_B(a,b) := \sigma_{a,b}\big(\{B\} \times M_B\big).$$

Partition the type space as follows:

$$R_A := \{(a,b) : p_A(a,b) > 0,\ p_B(a,b) = 0\},$$
$$R_B := \{(a,b) : p_B(a,b) > 0,\ p_A(a,b) = 0\},$$
$$R_{AB} := \{(a,b) : p_A(a,b) > 0,\ p_B(a,b) > 0\},$$
$$R_0 := \{(a,b) : p_A(a,b) = p_B(a,b) = 0\},$$

define the subsets:

$$R_A^1 := \{(a,b) \in R_A :\ p_A(a,b) = 1\},\ \ R_B^1 := \{(a,b) \in R_B :\ p_B(a,b) = 1\},$$
$$R_{AB}^1 := \{(a,b) \in R_{AB} :\ p_A(a,b) + p_B(a,b) = 1\},$$

and the residual subsets:

$$R_A^{<1} := R_A \setminus R_A^1, \qquad R_B^{<1} := R_B \setminus R_B^1, \qquad R_{AB}^{<1} := R_{AB} \setminus R_{AB}^1.$$

Note that:

$$\int_{R_A} p_A\, dF \le F(R_A), \quad \int_{R_B} p_B\, dF \le F(R_B), \quad \int_{R_{AB}} (p_A + p_B)\, dF \le \int_{R_{AB}} dF = F(R_{AB}).$$

Now, since $F$ has a density, for any Borel set $E \subset [0,1]^2$ and any $t \in [0, F(E)]$ there exists another Borel set $S \subset E$ with $F(S) = t$.[21] We apply this observation to the regions we defined in a way that ensures types without $\varnothing$ in $\sigma_{a,b}$ are not assigned $\varnothing$. For $R_{AB}^1$, choose $T_A \subset R_{AB}^1$ with

$$F(T_A) = \int_{R_{AB}^1} p_A\, dF,$$

and set $T_B := R_{AB}^1 \setminus T_A$. Next, choose sets on the residual regions:

$$S_A^{A,+} \subset R_A^{<1} \ \text{ with } \ F(S_A^{A,+}) = \int_{R_A^{<1}} p_A\, dF = \int_{R_A} p_A\, dF - F(R_A^1),$$

$$S_B^{B,+} \subset R_B^{<1} \ \text{ with } \ F(S_B^{B,+}) = \int_{R_B^{<1}} p_B\, dF = \int_{R_B} p_B\, dF - F(R_B^1),$$

$$S_A^{AB,+} \subset R_{AB}^{<1} \ \text{ with } \ F(S_A^{AB,+}) = \int_{R_{AB}^{<1}} p_A\, dF = \int_{R_{AB}} p_A\, dF - \int_{R_{AB}^1} p_A\, dF,$$

and then choose:

$$S_B^{AB,+} \subset R_{AB}^{<1} \setminus S_A^{AB,+} \ \text{ with } \ F(S_B^{AB,+}) = \int_{R_{AB}^{<1}} p_B\, dF = \int_{R_{AB}} p_B\, dF - \int_{R_{AB}^1} p_B\, dF.$$

[21] Let $f$ be the density of $F$. For $s \in [0,1]$, define $g(s) := F\big(E \cap ([0,s] \times [0,1])\big) = \int_0^s \int_0^1 \mathbf{1}_E(x,y) f(x,y)\, dy\, dx$. Note $g$ is continuous with $g(0) = 0$ and $g(1) = F(E)$, so we can pick $\big(E \cap ([0,s] \times [0,1])\big)$ with $s^*$ such that $g(s^*) = t$.

This last choice is feasible since on $R^{<1}_{AB}$ we have $p_A + p_B < 1$, hence $1 - p_A \geq p_B$, so:

$$F(R^{<1}_{AB} \setminus S^{AB,+}_A) = F(R^{<1}_{AB}) - F(S^{AB,+}_A) = \int_{R^{<1}_{AB}} (1 - p_A)\, dF \geq \int_{R^{<1}_{AB}} p_B\, dF = F(S^{AB,+}_B).$$

Finally, define:

$$S^A_A := R^1_A \cup S^{A,+}_A, \qquad S^B_B := R^1_B \cup S^{B,+}_B, \qquad S^{AB}_A := T_A \cup S^{AB,+}_A, \qquad S^{AB}_B := T_B \cup S^{AB,+}_B.$$

Now, let $\hat{y} : [0,1]^2 \to \{\varnothing, A, B\}$ satisfy:

$$\hat{y}(a,b) := \begin{cases} A & \text{if } (a,b) \in S^A_A \cup S^{AB}_A, \\ B & \text{if } (a,b) \in S^B_B \cup S^{AB}_B, \\ \varnothing & \text{otherwise,} \end{cases}$$

and define $\tilde{m} : [0,1]^2 \to \Omega$ as follows: if $\hat{y}(a,b) = A$, pick $\tilde{m}(a,b) \in \operatorname{supp}(\sigma_{a,b}) \cap (\{A\} \times M_A)$; if $\hat{y}(a,b) = B$, pick $\tilde{m}(a,b) \in \operatorname{supp}(\sigma_{a,b}) \cap (\{B\} \times M_B)$; and if $\hat{y}(a,b) = \varnothing$, set $\tilde{m}(a,b) = \varnothing$. Then, define the deterministic mechanism $\tilde{\sigma}_{a,b} := \delta_{\tilde{m}(a,b)}$. Moreover, note:

$$\begin{aligned} D_{\tilde{\sigma}}(\{A\} \times M_A) &= \int \tilde{\sigma}_{a,b}(\{A\} \times M_A)\, dF \\ &= \int \mathbf{1}_{\hat{y}=A}\, dF \\ &= F(S^A_A) + F(S^{AB}_A) \\ &= \int p_A\, dF = D_\sigma(\{A\} \times M_A), \end{aligned}$$

with an analogous expression holding for good $B$. Finally, fix any type $(a,b)$ and recall that all $m \in \operatorname{supp} \sigma_{a,b}$ gave this type the utility $u_{a,b}(m) = \max_{\omega \in \Omega} u_{a,b}(\omega)$. Since $\tilde{\sigma}$ assigns this type agent an option from her $\operatorname{supp} \sigma_{a,b}$, her utility under $\tilde{\sigma}$ is equal to that under $\sigma$. □

# C Omitted proofs

First, recall that $U_A, U_B$ are convex and strictly increasing on $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, respectively. As such, they are differentiable except at countably many points and are uniquely pinned down by their derivatives, wherever these exist, and $U_A(\underline{a}) = U_B(\underline{b}) = 0$. Moreover, wherever they fail to be differentiable, we have:

$$U'^{-}_A(t') = \lim_{t \uparrow t',\, t \in D} U'_A(t), \quad U'^{+}_A(t') = \lim_{t \downarrow t',\, t \in D} U'_A(t),$$

where $D \subset [\underline{a}, \overline{a}]$ is the set of points where $U'_A$ exists. By convention, we set $U'^{-}_A(0) = 0$, $U'^{+}_A(1) = 1$. Analogous statements hold for $U_B$.

### C.1 Proof of Proposition 1

Suppose some type $(a,b) < (\underline{a},\underline{b})$ could weakly benefit from requesting either good. Then some type $(a+\epsilon, b+\epsilon) < (\underline{a},\underline{b})$, for $\epsilon > 0$ small enough, would strictly benefit from it, so one of $U_A(a+\epsilon)$ and $U_B(b+\epsilon)$ would have to be strictly above zero. Since $U_A, U_B$ are increasing, this contradicts the definition of $(\underline{a},\underline{b})$. Thus, $y(a,b) = \varnothing$ for all $(a,b) < (\underline{a},\underline{b})$.

Analogously, all types for whom $a > \underline{a}$ or $b > \underline{b}$ strictly benefit from choosing one of the goods. Moreover, a positive mass of types gets either good, and thus $\underline{a}, \underline{b} < 1$.

Now, suppose w.l.o.g. that $U_B(1) \geq U_A(1)$. We will construct the boundary $z$ and prove it has the above properties. For $a \in [\underline{a}, 1]$, define:

$$z(a) := \inf\left\{b : U_B(b) \geq U_A(a)\right\}.$$

Note that, by construction, $z(\underline{a}) = \underline{b}$. Also, since $U_B(1) \geq U_A(1)$, it must be that $z(1) := \overline{b} \leq 1$.

Since $U_A$ and $U_B$ are both continuous and strictly increasing on $[\underline{a}, 1]$ and $[\underline{b}, 1]$, $z(a)$ is also continuous and satisfies:

$$U_A(a) = U_B(z(a)) \quad \Longrightarrow \quad z(a) = U_B^{-1}\left(U_A(a)\right).$$

Note $z(a)$ is a composition of two strictly increasing functions, so it is also strictly increasing.

By construction, any type $(a, z(a)) > (\underline{a},\underline{b})$ is indifferent between her best options for both goods. Then, by a single-crossing argument, any type $(a', z(a))$ with $a' > a$ strictly prefers to get good $A$. Analogously, any type $(a, b')$ with $b' > z(a)$ strictly prefers to get good $B$.

### C.2 Proof of Lemma 1

Proposition 1 lets us rewrite total welfare (W) in terms of $A,B$-indirect utilities $U_A, U_B$ and their associated extended boundary $\hat{z}$:

$$\int_{\underline{a}}^{1} \int_0^{\hat{z}(a)} f(a,v)dv \cdot U_A(a)\, da \; + \; \int_{\underline{b}}^{1} \int_0^{\hat{z}^{-1}(b)} f(v,b)dv \cdot U_B(b)\, db. \tag{W'}$$

Since $\overline{b} = 1$, the above becomes:

$$\int_{\underline{a}}^{\overline{a}} \int_0^{z(a)} f(a,v)dv \cdot U_A(a)\, da \; + \; \int_{z(\underline{a})}^{z(\overline{a})} \int_0^{z^{-1}(b)} f(v,b)dv \cdot U_B(b)\, db + \int_{\overline{a}}^{1} \int_0^{1} f(a,v)dv \cdot U_A(a)\, da.$$

A change of variables yields:

$$\int_{\underline{a}}^{\overline{a}} \underbrace{\left( \int_0^{z(a)} f(a,v)dv + \int_0^{a} f(v,z(a))dv \cdot z'(a) \right)}_{= \frac{d}{da} F(a,z(a))} U_A(a)\, da \; + \; \int_{\overline{a}}^{1} \underbrace{\int_0^{1} f(a,v)dv}_{= \frac{d}{da} F(a,1)} \; U_A(a)\, da.$$

Denote the former integral by $K$ and the latter one by $L$. Integrating them by parts gives:

$$K = U_A(\overline{a}) \cdot F(\overline{a}, z(\overline{a})) - U_A(\underline{a}) \cdot F(\underline{a}, z(\underline{a})) - \int_{\underline{a}}^{\overline{a}} U_A'(a) \cdot F(a, z(a))\, da.$$

Recall that $U_A(\underline{a}) = 0$. Thus, we get:

$$K = U_A(\overline{a}) \cdot F(\overline{a}, 1) - \int_{\underline{a}}^{\overline{a}} U_A'(a) \cdot F\left(a, z(a)\right)\, da.$$

Integrating $L$ by parts gives:

$$L = U_A(1) - U_A(\overline{a}) \cdot F(\overline{a}, 1) - \int_{\overline{a}}^{1} U_A'(a) \cdot F(a, 1)\, da.$$

Summing $K$ and $L$, we get:

$$U_A(1) - \int_{\underline{a}}^{\overline{a}} U_A'(a) \cdot F(a, z(a))\, da - \int_{\overline{a}}^{1} U_A'(a) \cdot F(a, 1)\, da = U_A(1) - \int_0^1 U_A'(a) \cdot F(a, \hat{z}(a)) da.$$

### C.3 Proof of Lemma 2

($\Rightarrow$). Fix any feasible mechanism $(c, x, y)$. I first show 1. Since $U_A$ and $U_B$ are convex and strictly positive above $\underline{a}, \underline{b}$, respectively, $U_A'$ and $U_B'$ are non-decreasing and strictly positive above $\underline{a}$ and $\underline{b}$. Moreover, Proposition 1 gives:

$$U_A(a) = U_B(z(a)) \quad \text{for all} \quad a \in [\underline{a}, \overline{a}]. \tag{16}$$

Differentiating gives the following wherever $z'$ exists:

$$U_A'(a) = U_B'(z(a)) \cdot z'(a) \quad \Rightarrow \quad U_B'(z(a)) = \frac{U_A'(a)}{z'(a)}.$$

Since $z$ is strictly increasing on $[\underline{a}, \overline{a}]$ and $U_B'$ is non-decreasing, $U_A'/z'$ is non-decreasing and strictly positive above $\underline{a}$. I now show 2. Recall that by Proposition 1, $z$ is pinned down by:

$$z = U_B^{-1} \circ U_A.$$

Fix any $a \in (\underline{a}, \overline{a}]$ and note that:

$$\begin{aligned}
z'^{+}(a) &= \lim_{\delta \downarrow 0} \frac{z(a+\delta) - z(a)}{\delta} \\
&= \lim_{\delta \downarrow 0} \frac{U_B^{-1}(U_A(a+\delta)) - U_B^{-1}(U_A(a))}{U_A(a+\delta) - U_A(a)} \cdot \frac{U_A(a+\delta) - U_A(a)}{\delta} \\
&= \left(U_B^{-1}\right)'^{+} (U_A(a)) \cdot U_A'^{+}(a) \\
&= \frac{U_A'^{+}(a)}{(U_B'^{+} \circ U_B^{-1} \circ U_A)(a)} = \frac{U_A'^{+}(a)}{(U_B'^{+} \circ z)(a)} > 0,
\end{aligned}$$

where the last inequality follows because $z = U_B^{-1} \circ U_A$ by (16). The argument for left derivatives is analogous. Let us now show 3 holds. Recall $U_A(a) > U_B(b)$ for almost all agents for whom $y(a,b) = A$, so by Proposition 1:

$$\int_{\underline{a}}^{1} \int_{0}^{\hat{z}(a)} f(a,v) dv \, da = \int \mathbb{1}_{y(a,b)=A} \, dF(a,b) \leq s_A,$$

where the inequality holds by (S). An analogous expression holds for $B$.

($\Leftarrow$). Fix any $(z, U_A)$ satisfying 1. – 3.; I construct a mechanism $(c, x, y)$ that corresponds to them. Choose some continuous $U_B : [0,1] \to [0,1]$ satisfying:

$$U_B'\big(z(a)\big) = \begin{cases} 0, & a < \underline{a}, \\ \dfrac{U_A'(a)}{z'(a)}, & a \in \big(\underline{a}, \overline{a}\big). \end{cases} \tag{17}$$

Note 1. guarantees that $U_B'$ is non-decreasing. I show that $U_A, U_B$ are the $A, B$-indirect utilities for the following mechanism, and that the mechanism is feasible:

$$c(a,b) = \begin{cases} a \cdot U_A'^{-}(a) - U_A(a), & \text{if } U_A(a) \geq U_B(b), \\ b \cdot U_B'^{-}(b) - U_B(b), & \text{if } U_B(b) > U_A(a), \end{cases}$$

$$x(a,b) = \begin{cases} U_A'^{-}(a), & \text{if } U_A(a) \geq U_B(b), \\ U_B'^{-}(b), & \text{if } U_B(b) > U_A(a), \end{cases}$$

$$y(a,b) = \begin{cases} \varnothing, & \text{if } (a,b) \leq (\underline{a}, \underline{b}), \\ A, & \text{if } a > \underline{a} \text{ and } U_A(a) \geq U_B(b), \\ B, & \text{if } b > \underline{b} \text{ and } U_B(b) > U_A(a). \end{cases}$$

Since $U_A, U_B$ are strictly increasing and convex, a standard argument verifies that, under this ordeal rule, no $(a,b)$ wants to misreport to $(a', b')$ for which $y(a',b') = y(a,b)$. That is, conditional on choosing the good she was assigned, $(a,b)$ prefers her assigned quality and ordeal option. Then $U_A, U_B$ are indeed the $A,B$-indirect utilities for this mechanism because:

$$a \cdot U_A'^{-}(a) - \left(a \cdot U_A'^{-}(a) - U_A(a)\right) = U_A(a), \qquad b \cdot U_B'^{-}(b) - \left(b \cdot U_B'^{-}(b) - U_B(b)\right) = U_B(b).$$

Verifying that (IC) holds for $(c,x,y)$ thus only requires checking that no $(a,b)$ wants to misreport to $(a',b')$ for which $y(a',b') \neq y(a,b)$. But since $U_A(a)$ and $U_B(b)$ are the best utilities $(a,b)$ can get from either good, this is true by the construction of $y(a,b)$. Note also that (IR) must hold as both $A,B$-indirect utilities are nonnegative everywhere.

I now verify that the mechanism implements boundary $z$. (17) tells us that wherever $z'$ exists, we have:

$$z'(a) \cdot U_B'(z(a)) = U_A'(a) \quad \Longrightarrow \quad \frac{d}{da} U_B(z(a)) = \frac{d}{da} U_A(a). \tag{18}$$

Moreover, $z$ is increasing, so $z'$ exists a.e. Since $U_A, U_B$ are absolutely continuous and $U_B(z(\underline{a})) = U_A(\underline{a}) = 0$, we can integrate both sides of (18) to get:

$$U_B(z(a)) = U_A(a).$$

It therefore remains to check the supply condition (S). But by Proposition 1:

$$\int \mathbb{1}_{y(a,b)=A} \, \mathrm{d}F(a,b) = \int_{\underline{a}}^{1} \int_0^{\hat{z}(a)} f(a,v)\, dv\, da \leq s_A,$$

$$\int \mathbb{1}_{y(a,b)=B} \, \mathrm{d}F(a,b) = \int_{\underline{b}}^{1} \int_0^{\hat{z}^{-1}(b)} f(v,b)\, dv\, db \leq s_B,$$

where the inequalities hold by 3.

### C.4 Lemma 2 under Assumption 2 and the proof of Fact 1

The following fact refines the characterization of feasible $(z, U_A)$ under Assumption 2. Fact 1 is a corollary of this result.

**Lemma 6.** *Let $z : [\underline{a}, \overline{a}] \to [\underline{b}, \overline{b}]$ be a boundary and $U_A$ be an $A$-indirect utility. Then, under Assumption 2, the pair $(z, U_A)$ is feasible if and only if:*

*(a) $U_A'$ and $U_A'/z'$ are non-decreasing and strictly positive above $\underline{a}$,*

*(b) The boundary $z$ has finite, strictly positive one-sided derivatives at every $a \in (\underline{a}, \overline{a})$, and a finite, strictly positive left derivative at $\overline{a}$.*

*(c) The supply constraint (S′) holds,*

*(d) $U_A', U_A'/z'$ are piecewise continuously differentiable and $z$ is piecewise twice continuously differentiable.*

*Proof.* The ($\Leftarrow$) direction follows because under $(a)-(d)$, Lemma 2 applies. Let us show the ($\Rightarrow$) direction. $(a)$, $(b)$ and $(c)$ follow from Lemma 2, so it remains to show $(d)$. By Proposition 1, types $(a,0)$ where $a>\underline{a}$ choose good $A$ and types $(0,b)$ where $b>\underline{b}$ choose good $B$. Then, by the envelope theorem, the following holds wherever $x(a,0)$, $x(0,b)$ are continuous in $a$, $b$, respectively:

$$U_A'(a)=x(a,0) \ \text{ for } \ a>\underline{a}, \quad U_B'(b)=x(0,b) \ \text{ for } \ b>\underline{b}. \tag{19}$$

Since $x$ is piecewise differentiable, this implies that $U_A'$ and $U_B'$ are piecewise continuously differentiable and $U_A, U_B$ are piecewise differentiable. Moreover, recall Proposition 1 tells us:

$$U_A=U_B\circ z \quad \Rightarrow \quad U_B^{-1}\circ U_A=z, \tag{20}$$

and thus $z$ is also piecewise continuously differentiable. Consider then any interval on which $z'$ exists. Differentiating the first equation in (20) then tells us that:

$$\frac{U_A'}{U_B'\circ z}=z'. \tag{21}$$

Then, since $U_A', U_B'$ and $z$ are all piecewise continuously differentiable, so is $z'$. We have thus shown $z$ is piecewise twice continuously differentiable and $U_A'$ is piecewise continuously differentiable. The fact that $U_A'/z'$ is piecewise continuously differentiable then follows from $U_A'$ and $z'$ being strictly positive and piecewise continuously differentiable on $[\underline{a},\overline{a}]$. □

### C.5 Proof of Lemma 3

The result follows from Proposition 7, which describes the solutions to a more general problem. To state it, let us call a boundary $\tilde{z}:[\underline{\tilde{a}},\overline{a}]\to[z(\underline{\tilde{a}}),\overline{b}]$ a *truncation* of boundary $z:[\underline{a},\overline{a}]\to[\underline{b},\overline{b}]$ if $\underline{\tilde{a}}\geq\underline{a}$ and:

$$z(a)=\tilde{z}(a) \quad \text{for } a\in[\underline{\tilde{a}},\overline{a}].$$

Note also that if boundary $z$ is implementable under Assumption 2, its truncation $\tilde{z}$ is implementable too: the conditions for implementability under Assumption 2 are given by Lemma 6, and truncation preserves the required monotonicity and differentiability properties, as well as relaxes the supply constraint.

**Problem 1.** *Fix an implementable boundary $z$. Choose a feasible pair $(\tilde{z},U_A)$ where $\tilde{z}$ is a truncation of $z$ to maximize:*

$$\int_0^1\Big(1-F(a,\hat{\tilde{z}}(a))-r\cdot\tilde{f}(a,\hat{\tilde{z}}(a))\Big)\cdot U_A'(a)\,da \tag{22}$$

*where $\hat{\tilde{z}}$ is the extended boundary for $\tilde{z}$, and:*

$$\tilde{f}(a,b)=\int_0^b f(a,t)dt+\int_0^a f(t,b)dt\cdot\mathbb{1}[b<1].$$

**Proposition 7.** *Under Assumption 2, Problem 1 has a solution* $(\tilde{z}, U_A^*)$ *such that:*

$$U_A^{*\prime}(a) := \begin{cases} 0, & a < \underline{a}, \\ m(a)\cdot k, & \underline{a} < a < \overline{a}, \\ m(\overline{a})\cdot k, & \overline{a} < a < h, \\ 1, & h < a, \end{cases} \quad \text{for some } h \geq \overline{a}, \tag{23}$$

*where:*

$$m(a) = \exp\left(\int_{\underline{a}}^{a} \max\left[0, \tfrac{z''(s)}{z'(s)}\right] ds\right) \prod_{\substack{z'^{+}(t) > z'^{-}(t),\\ t \leq a}} \frac{z'^{+}(t)}{z'^{-}(t)}, \qquad k = \frac{1}{\max[m(\overline{a}),\ m(\overline{a})/z'^{-}(\overline{a})]}.$$

**C.5.1 Proof of Proposition 7.** We first show the following lemma:

**Lemma 7.** *Let* $\mathcal{Y}$ *be the set of functions* $y : [0,1] \to [0,1]$ *such that:*

1. $y(a) = 0$ *on* $[0, \underline{a})$
2. $y(a)$ *is non-decreasing*
3. $y(a)/z'^{+}(a)$ *is non-decreasing on* $[0, \overline{a})$.
4. $y(a)/z'^{+}(a) \in [0,1]$.

*Then all extreme points of* $\mathcal{Y}$ *satisfy:*

$$y(a) := \begin{cases} 0, & a < l, \\ m(a)\cdot k, & l < a < \overline{a} \quad \text{if } l < \overline{a}, \\ m(\overline{a})\cdot k, & \overline{a} < a < h, \\ 1, & h < a, \end{cases} \quad \text{for some } l, h \text{ such that } h \geq \overline{a},\ \underline{a} \leq l \leq h \leq 1. \tag{24}$$

Note that the lemma does not characterize the set of extreme points, but only provides a necessary condition for $y$ being an extreme point.

*Proof.* First, consider any $y \in \mathcal{Y}$ that does not satisfy (24) on $[\overline{a}, 1]$. Then the standard characterization of extreme points of monotone functions (Choquet, 1954) implies $y$ can be written as a convex combination of $y_h \in \mathcal{Y}$ of the form:

$$y_h(a) := \begin{cases} y(a), & a < \overline{a}, \\ y(\overline{a}), & \overline{a} \leq a < h, \\ 1, & h < a. \end{cases} \quad \text{for some } h \geq \overline{a}. \tag{25}$$

Moreover, note that any extreme point of $\mathcal{Y}$ must satisfy $y(h) \geq m(a) \cdot k$. Otherwise, we could find $\epsilon > 0$ small enough that $y \cdot (1-\epsilon),\ y \cdot (1+\epsilon) \in \mathcal{Y}$ and write $y$ as their convex combination.

Thus, we can restrict the search for extreme points to functions $y \in \mathcal{Y}$ that satisfy:

$$y(a) = \begin{cases} y(\overline{a}), & \overline{a} \leq a < h, \\ 1, & h < a. \end{cases} \qquad \text{for some } h \geq \overline{a}, \text{ with } y(h) \geq m(a) \cdot k. \tag{26}$$

It then suffices to show that all such $y$ can be written as a convex combination of functions from $\mathcal{Y}$ satisfying (24). To that end, we prove the following fact:

**Fact 2.** *For any $y \in \mathcal{Y}$ satisfying (26), the ratio $\frac{y(a)}{m(a) \cdot k}$ is non-decreasing on $[\underline{a}, \overline{a}]$.*

*Proof.* Let $\hat{a} := \inf\{a \geq \underline{a} : y(a) > 0\}$. For $a < \hat{a}$, we have $y(a) = 0$ and hence the ratio is zero there. Since the ratio is always non-negative, it suffices to show that it is non-decreasing on $[\hat{a}, \overline{a}]$.

We now show monotonicity on $(\hat{a}, \overline{a})$. Since $z$ is piecewise twice continuously differentiable, we can decompose $(\hat{a}, \overline{a})$ into a disjoint union of open intervals on which $z$ is $C^2$ and a finite set of points where $z'$ has jumps. Let us first consider an interval $(a_1, a_2)$ on which $z$ is $C^2$. There it suffices to show that $\log(\frac{y}{m \cdot k})$ is non-decreasing. Note that:

$$\begin{aligned} \log\left(\frac{y}{m \cdot k}\right) &= \log(y) - \log(m \cdot k) \\ &= \text{const} + \log(y) - \int_{a_1}^{a} \max\left[0, \frac{z''(t)}{z'(t)}\right] dt. \end{aligned}$$

We now use the following observation:

**Observation 1.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $g$ is continuous and piecewise continuously differentiable, and $f$ and $f - g$ are non-decreasing. Then the following function is non-decreasing in $a$:*[22]

$$f(a) - \int_{a_1}^{a} \max[0, g'(t)]\, dt.$$

Since $y \in \mathcal{Y}$, we know that $\log(y)$ and $\log(y) - \log(z')$ are both non-decreasing. Applying Observation 1 then tells us that the following function is non-decreasing in $a$:

$$\log(y(a)) - \int_{a_1}^{a} \max\left[0, (\log z'(t))'\right] dt = \log(y(a)) - \int_{a_1}^{a} \max\left[0, \frac{z''(t)}{z'(t)}\right] dt.$$

We now check points where $z'$ has jumps. There we need to show that:

$$\left(\frac{y}{m \cdot k}\right)^{+} \geq \frac{y}{m \cdot k} \geq \left(\frac{y}{m \cdot k}\right)^{-}. \tag{27}$$

[22] By convention, let us set $\max[0, g'(t)] = 0$ wherever $g'(t)$ does not exist.

The first inequality follows by monotonicity of $y$ and the fact that $m$ is right-continuous. It thus remains to show that:

$$\frac{y}{m \cdot k} \geq \left(\frac{y}{m \cdot k}\right)^{-}. \tag{28}$$

Consider first points where $z'$ jumps upwards. Note $\left(\frac{y}{m \cdot k}\right)^{-} = \frac{y^{-}}{m^{-} \cdot k}$ and that:

$$m \cdot k = m^{+} \cdot k = m^{-} \cdot k \cdot \frac{z'^{+}}{z'^{-}} \quad \Rightarrow \quad \frac{m \cdot k}{m^{-} \cdot k} = \frac{z'^{+}}{z'^{-}}.$$

Thus, (28) is equivalent to:

$$\frac{y}{y^{-}} \geq \frac{z'^{+}}{z'^{-}}.$$

But since $y/z'^{+}$ is non-decreasing, we have:

$$\frac{y}{z'^{+}} \geq \lim_{a' \to a^{-}} \left(\frac{y(a')}{z'(a')}\right) = \frac{y^{-}}{z'^{-}} \quad \Rightarrow \quad \frac{y}{y^{-}} \geq \frac{z'^{+}}{z'^{-}}.$$

Now consider points where $z'$ jumps downwards. By the definition of $m$ we know that $m \cdot k$ is continuous there. Since $y$ is non-decreasing, it is either continuous or jumps upwards at such points. In either case, we get (28).

It therefore remains to verify that monotonicity is preserved at $\hat{a}$ and $\overline{a}$. If $\hat{a} = \underline{a}$, this follows by monotonicity of $y$ and the right-continuity of $m$. If $\hat{a} > \underline{a}$, we have two cases. If $m$ is continuous at $\hat{a}$, this follows from the monotonicity of $y$. If $m$ is discontinuous there, $\hat{a}$ has to be a point where $z'$ jumps upwards, in which case the argument is analogous to the jump case above. Finally, monotonicity at $\overline{a}$ follows because $m$ is left-continuous there and $y$ is non-decreasing. □

Let us now define a probability measure $\mu$ over $[\underline{a}, h]$ such that:

$$\mu\big([\underline{a}, a]\big) := \begin{cases} \frac{y(a)}{m(a) \cdot k}, & \underline{a} \leq a < \overline{a}, \\ \frac{y(\overline{a})}{m(\overline{a}) \cdot k}, & \overline{a} \leq a < h, \\ 1, & a = h. \end{cases}$$

To see why $\mu$ is indeed a probability measure, note first that $\mu\left([\underline{a}, a]\right)$ is non-decreasing in $a$ for $a \in [\underline{a}, h)$. Indeed, Fact 2 guarantees the monotonicity of $\frac{y(a)}{m(a) \cdot k}$ on $[\underline{a}, \overline{a}]$. Since $\mu\left([\underline{a}, a]\right) = \frac{y(\overline{a})}{m(a) \cdot k}$ for $a \in [\overline{a}, h)$, monotonicity is also guaranteed there. Moreover, the construction of $m(a) \cdot k$ guarantees that $m(\overline{a}) \cdot k \geq y(\overline{a})$ for all $y \in \mathcal{Y}$, and therefore $\frac{y(a)}{m(a) \cdot k} \in [0, 1]$.

Now, let $y_{l,h}$ denote functions satisfying (24) that are right-continuous on $[0, h)$ and such that $y_{l,h}(h) = y(h)$.[23] We can then write $y$ as a $\mu$-convex combination of $y_{l,h}$:

$$y = \int y_{l,h} \, d\mu(l). \tag{29}$$

[23]One can easily verify that $y_{l,h}$ defined this way belong to $\mathcal{Y}$.

Indeed, note that:

$$\int y_{l,h}(a)\,d\mu(l) = \begin{cases} 0 = y(a), & a < \underline{a}, \\ m(a)\cdot k\cdot \mu\big([\underline{a},a]\big) = y(a), & a \in [\underline{a},\overline{a}), \\ m(\overline{a})\cdot k\cdot \mu\big([\underline{a},a]\big) = y(a), & a \in [\overline{a},h), \\ y_{l,h}(h)\cdot \mu\big([\underline{a},h]\big) = y_{l,h}(h) = y(h), & a = h. \end{cases}$$

□

We are now ready to prove Proposition 7. We will say that $y \in \mathcal{Y}$ corresponds to $U_A$ if, wherever $U'_A$ exists, we have:

$$y(a) = U'_A(a).$$

Note that for every feasible $U_A$ there exists the following corresponding $y$:

$$y(a) = \begin{cases} U'_A(a), & \text{if } U'_A(a) \text{ exists}, \\ U'^{+}_A(a), & \text{otherwise.} \end{cases} \tag{30}$$

Indeed, Lemma 6 tells us that an $A$-indirect utility function $U_A$ is feasible if and only if $U'_A$ and $U'_A/z'$ are non-decreasing, piecewise continuously differentiable and strictly positive above $\underline{a}$, which implies that $y$ given by (30) belongs to $\mathcal{Y}$. Also, since $y(a) = U'_A(a)$ almost everywhere, the value of the objective (22) is the same for $U_A$ and its corresponding $y$.

Let us then consider the following problem:

**Problem 2.** *Choose $y \in \mathcal{Y}$ to maximize (22).*

We have shown that for every feasible $U_A$ there exists a corresponding $y$ which gives an equal value of (22). Thus, the value of Problem 2 is higher than that of Problem 1. Consequently, if $y^*$ solves Problem 2 and has a corresponding $U_A$ such that $(\tilde{z}, U_A)$ is feasible for some $\tilde{z}$, then this pair also solves Problem 1.

I now show there exists an extreme point $\hat{y}^*$ that solves Problem 2 and corresponds to some feasible $(\tilde{z}^*, U^*_A)$. Then this pair solves Problem 1 and $U^*_A$ satisfies (23), which is what we want to show. To show this, let us first note that $\mathcal{Y}$ is convex, compact and the objective (22) is linear, and so there exists a solution to Problem 2 that is an extreme point of $\mathcal{Y}$ (Phelps, 2001). Let $\hat{y}^*$ be such an extremal solution. By Lemma 7, it satisfies (24) for some $l, h \geq \underline{a}$. Now, let $\tilde{z}$ be a truncation of $z$ such that $\underline{\tilde{a}} = l$ and observe that $\hat{y}^*$ corresponds to some $U_A$ such that $(\tilde{z}, U_A)$ is feasible. This is because (24) tells us $\hat{y}^*$, $\hat{y}^*/z'$ are both non-decreasing and piecewise continuously differentiable, and that $\hat{y}^*$ is strictly positive above $l$ and zero below it.

**C.5.2 Proof of Lemma 3.** Fix any boundary $z$ implementable under Assumption 2 and consider Problem 1 for this boundary and $r = 0$. Note that if the problem is solved by a pair $(z, U^*)$ featuring the boundary $z$ itself, then $U^*_A$ must be optimally implementing this boundary.

By Proposition 7, there exists a solution $(\tilde{z}^*, U_A^*)$ to this problem such that $U_A^*$ satisfies (23). Note that among all pairs $(\tilde{z}, U_A)$ that satisfy (23), the one featuring $z$ and $U_A$ satisfying (7) is the one with the uniquely pointwise highest $U_A'$. Now, since $r = 0$, the objective puts strictly positive weight on the value of $U_A'$ everywhere, and thus the extremal solution must feature this pointwise highest $U_A$.

### C.6 Proof of Proposition 2

**The optimal boundary is piecewise linear.** I begin by showing that $z^*$ has to solve the following optimal control problem on every closed interval where it is concave and twice continuously differentiable:

**Problem 3.** *Choose the control $u : [\underline{v}, \overline{v}] \to \mathbb{R}_-$ and state variables $z, y, q : [\underline{v}, \overline{v}] \to \mathbb{R}$ to maximize:*

$$-\int_{\underline{v}}^{\overline{v}} F\left(a, z(a)\right) da, \tag{31}$$

*subject to the following laws of motion:*

$$z'(v) = y(v), \quad y'(v) = u(v), \quad q'(v) = \int_0^{z(v)} f(v,b)db,$$

*and the following end-point constraints:*

$$z(\underline{v}) = z^*(\underline{v}), \quad z(\overline{v}) = z^*(\overline{v}), \tag{32}$$

$$y(\underline{v}) = z_+^{*\prime}(\underline{v}), \quad y(\overline{v}) = z_-^{*\prime}(\overline{v}), \tag{33}$$

$$q(\underline{v}) = 0, \quad q(\overline{v}) = \int_{\underline{v}}^{\overline{v}} \left( \int_0^{z^*(v)} f(v,b)db \right) dv. \tag{34}$$

The states $z$ and $y$ correspond to the boundary and its derivative, the control $u$ corresponds to its second derivative, and $q$ is introduced to capture the supply constraint.

**Lemma 8.** *Let $[\underline{v}, \overline{v}]$ with $\underline{v} > \underline{a}$ be such that $z^*$ is twice continuously differentiable with $z^{*\prime\prime} < 0$ on it. Then $z^*$ has to solve Problem 3 on $[\underline{v}, \overline{v}]$.*

*Proof.* First, note that by Fact 1 $z^*$ is absolutely continuous on $[\underline{v}, \overline{v}]$. It is also concave and twice continuously differentiable on this interval, so it is admissible in Problem 3. By Lemma 3, the optimal $U_A$ for $z^*$ satisfies the following on $[\underline{v}, \overline{v}]$:

$$U_A'(a) = k,$$

for some $k > 0$. Now, take any $z : [\underline{v}, \overline{v}] \to \mathbb{R}$ that is admissible in Problem 3 and consider:

$$\tilde{z}(v) = \begin{cases} z(v) & \text{if } v \in [\underline{v}, \overline{v}], \\ z^*(v) & \text{elsewhere.} \end{cases}$$

Note that, by Lemma 6, the pair $(\tilde{z}, U_A)$ is also implementable. Thus, if $z^*$ is optimal globally, it must be optimal among all $\tilde{z}$ implemented this way. In what follows I show that this is only true if $z^*$ solves Problem 3. Indeed, by Lemma 1, total welfare is:

$$U_A(1) - \int_0^1 U_A'(a) \cdot F\left(a, \hat{z}(a)\right) da = \text{const} - k \cdot \int_{\underline{v}}^{\overline{v}} F\left(a, z(a)\right) da.$$

□

I now show $z^{*\prime\prime} = 0$ wherever it exists. Suppose $z^{*\prime\prime} < 0$ somewhere (the case of $z^{*\prime\prime} > 0$ is symmetric). By Fact 1, $z^{*\prime\prime}$ is piecewise continuous, so there must exist an interval $[\underline{v}, \overline{v}]$, with $\underline{v} > \underline{a}$, on which $z^{*\prime\prime}$ exists and $z^{*\prime\prime} < 0$.

Consider Problem 3 for that interval. As shown, $z^*$ restricted to $[\underline{v}, \overline{v}]$ must be the optimal $z$ for that problem. Let $(z^*, y^*, q^*, u^*, \xi, \phi, \mu)$ be the optimal collection of states, controls, and costates associated with $z^*$. The Hamiltonian for this problem is:

$$\mathcal{H} = -F\left(a, z^*(a)\right) + \mu(a) \cdot \left( \int_0^{z^*(a)} f(a,b) db \right) + \xi(a) \cdot y(a) + \phi(a) \cdot u(a), \tag{35}$$

where $\mu(a)$ is the costate on $q$, $\xi$ is the costate on $z$ and $\phi$ is the costate on $y$. By the Maximum Principle, we then have:

$$\mu'(a) = 0.$$

Since $\mu(a)$ is constant, I simply write it as $\mu$. Moreover, we have:

$$\xi'(a) = -\left( - \int_0^a f\left(v, z^*(a)\right) dv + \mu \cdot f\left(a, z^*(a)\right) \right) = \int_0^a f\left(v, z^*(a)\right) dv - \mu \cdot f\left(a, z^*(a)\right), \tag{36}$$

and:

$$\phi'(a) = -\xi(a), \tag{37}$$

giving:

$$\phi''(a) = -\xi'(a) = -\left( \int_0^a f\left(v, z^*(a)\right) dv - \mu \cdot f\left(a, z^*(a)\right) \right). \tag{38}$$

The Maximum principle further tells us that controls $u^*(v) < 0$ must maximize the Hamiltonian everywhere in $(\underline{v}, \overline{v})$. However, the Hamiltonian depends on the control linearly and so the optimal control can be interior only if $\phi(v) = 0$ on $(\underline{v}, \overline{v})$. In particular, this means that $\phi''(a)$ has to be zero in that region. This gives:

$$0 = -\int_0^a f\left(v, z^*(a)\right) dv + \mu \cdot f\left(a, z^*(a)\right) \quad \Rightarrow \quad \frac{\int_0^a f\left(v, z^*(a)\right) dv}{f\left(a, z^*(a)\right)} = \frac{F_{A|B}\left(a|z^*(a)\right)}{f_{A|B}\left(a|z^*(a)\right)} = \mu. \tag{39}$$

Note, however, that $z^*(a)$ is strictly increasing in $a$ and, by Assumption 1, the inverse conditional anti-hazard rate is strictly increasing in one of $a$ and $z^*$, and non-decreasing in the other. Thus, (39) cannot hold on an interval and so $z^{*\prime\prime}(a) = 0$ wherever $z^*$ is twice-differentiable. Since $z^*$ was piecewise twice continuously differentiable, it follows that $z^*$ is piecewise linear.

**The optimal boundary is linear.** We know the optimal boundary $z^*$ is piecewise linear. Pick $\overline{v}$ such that $[\underline{a}, \underline{v}]$ is the largest interval starting with $\underline{a}$ on which $z^*$ is convex or concave. Assume w.l.o.g. that $z^*$ is concave on it and consider the following problem:

**Problem 4.** *Choose the control $u : [\underline{a}, \overline{v}] \to \mathbb{R}_-$, state variables $z, y, q : [\underline{a}, \overline{v}] \to \mathbb{R}$, a number of jumps $n \in \mathbb{N}$, jump locations and jump sizes, $a_i \in [\underline{a}, \overline{v}]$ and $v_i \in \mathbb{R}_-$ for $i \in \{1, \ldots, n\}$ to maximize:*

$$-\int_{\underline{a}}^{\overline{v}} F\left(a, z(a)\right) da. \tag{40}$$

*subject to the following laws of motion:*

$$z'(v) = y(v), \quad y'(v) = u(v), \quad q'(v) = \left(\int_0^{z(v)} f(v,b)db\right),$$

*the jump function for every $i$:*

$$v_i = y_+(a_i) - y_-(a_i),$$

*and the following end-point constraints:*

$$z(\underline{a}) = z^*(\underline{a}), \quad z(\overline{v}) = z^*(\overline{v}), \tag{41}$$

$$y(\underline{a}) \text{ free}, \quad y(\overline{v}) = z^{*\prime}_-(\overline{v}), \tag{42}$$

$$q(\underline{a}) = 0, \quad q(\overline{v}) = \int_{\underline{a}}^{\overline{v}} \left(\int_0^{z^*(v)} f(v,b)db\right) dv. \tag{43}$$

We then get the following lemma whose proof is analogous to that of Lemma 8:

**Lemma 9.** *Let $[\underline{a}, \overline{v}]$ be a concave interval of $z^*$. Then the optimal boundary $z^*$ on this interval has to solve Problem 4.*

Since $z^*$ is piecewise linear, the largest initial concave interval either covers all of $[\underline{a}, \overline{a}]$, or consists of at least two linear pieces. In what follows, I show that the solution to Problem 4 cannot have jumps, and thus that the latter cannot happen. This in turn proves that $z^*$ is linear.

Let us now analyze the necessary conditions for $z^*$ restricted to $[\underline{a}, \overline{v}]$ to solve Problem 4. The Hamiltonian and costate equations for this problem are the same as for Problem 3, and given by (35), (36) and (37), and $\mu$, the costate for $q$, is also constant. However, the initial value of $y$ is now free, so its costate at the beginning of the interval is zero (see Neustadt (1976), p. 234):

$$\phi(\underline{a}) = 0. \tag{44}$$

Now, by the Maximum Principle with jumps (see Seierstad and Sydsaeter (1986), Theorem 7, p. 196-197) we know that:

1. $\phi(\cdot)$ is continuous and differentiable except possibly at jump points,
2. $\phi(a^*) = 0$ when $a^*$ is a jump point,

3. At all $a$ where there is no jump, $\phi(a) \geq 0$.

**Fact 3.** *The costate $\phi$ is twice continuously differentiable on $(\underline{a},\overline{v})$.*

*Proof.* For $a$ other than jump points this follows because:

$$\phi'(a) = -\xi(a) = -\xi(\underline{a}) - \int_{\underline{a}}^{a} \xi'(t)dt = -\xi(\underline{a}) - \int_{\underline{a}}^{a} \left( \int_0^t f\left(v,z^*(t)\right) dv - \mu \cdot f\left(t,z^*(t)\right) \right) dt. \quad (45)$$

Now, let $a^* \in (\underline{a},\overline{v})$ be a jump point. Then (45) holds on some open neighborhoods to the left and right of $a^*$. We know that $\phi'(a)$ is differentiable there, with:

$$\phi''(a) = -\xi'(a) = -\left( \int_0^a f\left(v,z^*(a)\right) dv - \mu \cdot f\left(a,z^*(a)\right) \right),$$

which, just like $\phi'(a)$, inherits continuity from $f$ and $F$. Moreover, we see that:

$$\lim_{a\downarrow a^*} \phi'(a) = \lim_{a\uparrow a^*} \phi'(a), \quad \lim_{a\downarrow a^*} \phi''(a) = \lim_{a\uparrow a^*} \phi''(a),$$

where these limits are finite. Thus, $\phi'(a^*), \phi''(a^*)$ also exist and equal to these limits, and so $\phi$ is indeed twice continuously differentiable on $(\underline{a},\overline{v})$. □

Point 2 above tells us that if there is an interior jump at $a^* \in (\underline{a},\overline{v})$, we must have $\phi(a^*) = 0$ there. Point 3 tells us that $\phi(a) \geq 0$ outside of jump points. Since, by Fact 3, $\phi$ is twice continuously differentiable around $a^*$, we must therefore have $\phi'(a^*) = 0$ and $\phi''(a^*) \geq 0$ there. I show this cannot happen, and thus that $z^*$ cannot have jumps. Note that:

$$\phi''(a) \;=\; f\left(a,z^*(a)\right) \left( \mu - \frac{\int_0^a f\left(v,z^*(a)\right) dv}{f\left(a,z^*(a)\right)} \right) \;=\; f\left(a,z^*(a)\right) \left( \mu - \frac{F_{A|B}\left(a|z^*(a)\right)}{f_{A|B}\left(a|z^*(a)\right)} \right).$$

Recall that $z^*(a)$ is strictly increasing in $a$, so by Assumption 1, the inverse conditional anti-hazard rate is strictly increasing in $a$. Thus, $\phi''(a)$ is either strictly negative everywhere on $(\underline{a},\overline{v})$ or positive until some $\tilde{a} \in (\underline{a},\overline{v})$ and then negative forever after. In the former case, $\phi''(a) < 0$ for all $a \in (\underline{a},\overline{v})$, and so $\phi''(a^*) \geq 0$ can never hold for an interior $a^*$. Let us then consider the latter case in which there exists some $\tilde{a} \in (\underline{a},\overline{v})$ such that:

$$\phi''(a) \begin{cases} > 0, & \text{if } a < \tilde{a}, \\ = 0, & \text{if } a = \tilde{a}, \\ < 0, & \text{if } a > \tilde{a}. \end{cases}$$

Now, I show that for every $a \in (\underline{a},\tilde{a}]$ we have $\phi'(a) > 0$. For suppose $\phi'(a) \leq 0$ for some $a \in (\underline{v},\tilde{a}]$. Then, since $\phi''(a) > 0$ on $(\underline{a},\tilde{a})$, it must be that $\phi'(a) < 0$ everywhere before $a$. Then, however:

$$\phi(a) = \underbrace{\phi(\underline{a})}_{=0} + \int_{\underline{a}}^{a} \underbrace{\phi'(v)}_{<0} dv < 0,$$

which cannot be as $\phi(a) \geq 0$ on the whole interval by the Maximum Principle with jumps. Thus, $\phi'(a) > 0$ for $a \in (\underline{a}, \tilde{a}]$ and $\phi''(a) < 0$ for all $a \in (\tilde{a}, \overline{v})$. Therefore there is no $a^* \in (\underline{a}, \overline{a})$ for which $\phi'(a^*) = 0$ and $\phi''(a^*) \geq 0$.

### C.7 Proof of Lemma 4

Fix any linear boundary $\hat{z}$. Throughout the proof we show that if the slope of $\hat{z}$ is not 1, it can be improved by changing the slope in the direction of 1 while preserving the total amounts of both goods allocated. To that end, let $z_s : [\underline{a}_s, \overline{a}_s] \to [\underline{b}_s, \overline{b}_s]$ denote a linear boundary with slope $s > 0$ under which the same amounts of $A$ and $B$ are allocated as under $\hat{z}$. Thus:

$$z_s(a) = \underline{b}_s + s \cdot (a - \underline{a}_s) \quad \text{for } a \in [\underline{a}_s, \overline{a}_s].$$

See Figure 14a. Note $\underline{a}_s$ and $\underline{b}_s$ are uniquely pinned down by the requirement that probability masses above and below $z_s$ match those above and below $\hat{z}$. Note that for every $s_1, s_2 > 0$ such that $s_1 > s_2$, the boundaries $z_{s_1}$ and $z_{s_2}$ cross exactly once, with $z_{s_2}$ crossing from above (Figure 14b). This in turn implies that $\underline{a}_s, \overline{b}_s$ increase in $s$ and $\overline{a}_s, \underline{b}_s$ decrease in $s$. Moreover, the regularity of the density $f$ guarantees that all four end-points change continuously in $s$.

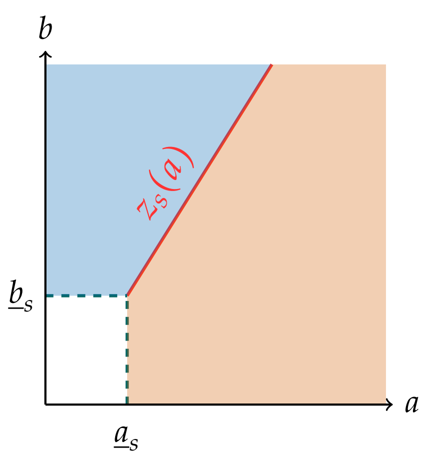


Figure 14a: $s$-sloped boundary $z_s$.

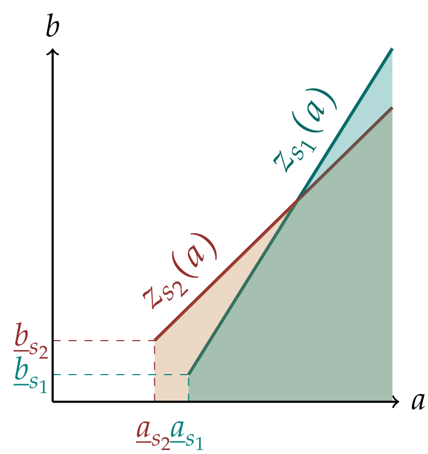


Figure 14b: $z_{s_2}$ crosses $z_{s_1}$ once, and from above.

We will take $W[z_s]$ to mean welfare under $(z_s, U_A)$, where $U_A$ is optimal for $z_s$. It thus suffices to show that $W[z_s]$ decreases in $s$ when $s > 1$ and increases in $s$ when $s < 1$. We show this using the following auxiliary fact:

**Fact 4.** *The following expression is strictly increasing in $s$ for $s > 1$:*

$$\int_0^1 F\left(a, \hat{z}_s(a)\right)\, da.$$

*Proof.* First, note that

$$F\big(a,\hat{z}_{s_1}(a)\big) - F\big(a,\hat{z}_{s_2}(a)\big) = \int_0^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)\,dv\,db \;-\; \int_0^{\hat{z}_{s_2}(a)} \int_0^a f(v,b)\,dv\,db$$
$$= \int_{\hat{z}_{s_2}(a)}^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)\,dv\,db.$$

Now, consider the difference:

$$\int_0^1 F\left(a,\hat{z}_{s_1}(a)\right)\,da - \int_0^1 F\left(a,\hat{z}_{s_2}(a)\right)\,da = \int_0^1 F\left(a,\hat{z}_{s_1}(a)\right) - F\left(a,\hat{z}_{s_2}(a)\right)da$$
$$= \int_0^1 \left(\int_{\hat{z}_{s_2}(a)}^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)dv\,db\right) da.$$

Since $s_1 > s_2$, $\hat{z}_{s_2}$ crosses $\hat{z}_{s_1}$ only once, and from above. Let $a^* \in (0,1)$ be their crossing point and define:

$$\underline{\mathcal{D}} = \left\{(a,b):\ 0 < a < a^*,\ \hat{z}_{s_1}(a) < b < \hat{z}_{s_2}(a)\right\}, \quad \overline{\mathcal{D}} = \left\{(a,b):\ a^* < a < 1,\ \hat{z}_{s_2}(a) < b < \hat{z}_{s_1}(a)\right\}.$$

We can then write the difference as:

$$\int_0^1 F\left(a,\hat{z}_{s_1}(a)\right)\,da - \int_0^1 F\left(a,\hat{z}_{s_2}(a)\right)\,da$$
$$= \int_{a^*}^1\left(\int_{\hat{z}_{s_2}(a)}^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)\,dv\,db\right)da \;+\; \int_0^{a^*}\left(\int_{\hat{z}_{s_2}(a)}^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)\,dv\,db\right)da$$
$$= \int_{a^*}^1\left(\int_{\hat{z}_{s_2}(a)}^{\hat{z}_{s_1}(a)} \int_0^a f(v,b)\,dv\,db\right)da \;-\; \int_0^{a^*}\left(\int_{\hat{z}_{s_1}(a)}^{\hat{z}_{s_2}(a)} \int_0^a f(v,b)\,dv\,db\right)da$$
$$= \int_{\overline{\mathcal{D}}}\left(\int_0^a f(v,b)\,dv\right)\mathrm{d}(a,b) \;-\; \int_{\underline{\mathcal{D}}}\left(\int_0^a f(v,b)\,dv\right)\mathrm{d}(a,b)$$
$$= \int_{\overline{\mathcal{D}}} \frac{\int_0^a f(v,b)\,dv}{f(a,b)}\cdot f(a,b)\,\mathrm{d}(a,b) \;-\; \int_{\underline{\mathcal{D}}} \frac{\int_0^a f(v,b)\,dv}{f(a,b)}\cdot f(a,b)\,\mathrm{d}(a,b)$$
$$= \int_{\overline{\mathcal{D}}} \frac{F_{A|B}(a\mid b)}{f_{A|B}(a\mid b)}\cdot f(a,b)\,\mathrm{d}(a,b) \;-\; \int_{\underline{\mathcal{D}}} \frac{F_{A|B}(a\mid b)}{f_{A|B}(a\mid b)}\cdot f(a,b)\,\mathrm{d}(a,b).$$

Now, recall that the inverse conditional anti-hazard rate is strictly increasing in $a$ and non-decreasing in $b$. Let $b^* := \hat{z}_{s_1}(a^*) = \hat{z}_{s_2}(a^*)$ and notice that, by single-crossing:

$$(a,b) > (a^*,b^*) \quad \text{for all } (a,b) \in \overline{\mathcal{D}}, \qquad (a,b) < (a^*,b^*) \quad \text{for all } (a,b) \in \underline{\mathcal{D}}.$$

Thus, we can bound the difference from below as follows:

$$\begin{aligned}
&\int_0^1 F\left(a, \hat{z}_{s_1}(a)\right)\, da - \int_0^1 F\left(a, \hat{z}_{s_2}(a)\right)\, da \\
&> \int_{\overline{\mathcal{D}}} \frac{F_{A|B}(a^* \mid b^*)}{f_{A|B}(a^* \mid b^*)} f(a,b)\,\mathrm{d}(a,b) \; - \; \int_{\underline{\mathcal{D}}} \frac{F_{A|B}(a^* \mid b^*)}{f_{A|B}(a^* \mid b^*)} f(a,b)\,\mathrm{d}(a,b) \\
&= \frac{F_{A|B}(a^* \mid b^*)}{f_{A|B}(a^* \mid b^*)} \left( \int_{\overline{\mathcal{D}}} f(a,b)\,\mathrm{d}(a,b) - \int_{\underline{\mathcal{D}}} f(a,b)\,\mathrm{d}(a,b) \right) = 0.
\end{aligned}$$

The difference is zero because the probability masses underneath both extended boundaries were equal. □

Recall that by Lemma 1, total welfare equals to:

$$U_A(1) - \int_0^1 U_A'(a) \cdot F\left(a, \hat{z}(a)\right) da. \tag{46}$$

Now, consider two cases.

**Case 1:** $s > 1$. In this case, the optimal $U_A$ for $z_s$ satisfies:

$$U_A'(a) = \begin{cases} 0, & \text{if } a < \underline{a}_s, \\ 1, & \text{if } a > \underline{a}_s. \end{cases}$$

Thus, total welfare is:

$$W[z_s] = 1 - \underline{a}_s - \int_0^1 F\left(a, \hat{z}_s(a)\right)\, da.$$

Now, since $\overline{a}_s, \overline{b}_s$ and $\underline{a}_s$ change continuously in $s$ and $\overline{a}_{s_1} < 1$, there exists $s_2 \in (1, s_1)$ such that $\overline{a}_{s_2} < 1$. Then also $\overline{b}_{s_2} = 1$ and thus we can apply the formula derived above to both $z_{s_1}$ and $z_{s_2}$:

$$W[z_{s_1}] = 1 - \underline{a}_{s_1} - \int_0^1 F\left(a, \hat{z}_{s_1}(a)\right)\, da, \quad W[z_{s_2}] = 1 - \underline{a}_{s_2} - \int_0^1 F\left(a, \hat{z}_{s_2}(a)\right)\, da.$$

It thus suffices to show that $W[z_{s_1}] < W[z_{s_2}]$. However, $s_1 > s_2$, so $\underline{a}_{s_1} > \underline{a}_{s_2}$. Moreover, Fact 4 tells us that

$$\int_0^1 F\left(a, \hat{z}_{s_1}(a)\right)\, da > \int_0^1 F\left(a, \hat{z}_{s_2}(a)\right)\, da,$$

which completes the proof.

**Case 2:** $s < 1$. Since $U_A'(a) = 1$ for $a > \overline{a}_s$, we can write total welfare as:

$$W[z_s] = \int_{\underline{a}_s}^{\overline{a}_s} U_A'(a) \cdot \left[1 - F\left(a, z(a)\right)\right] da + \int_{\overline{a}_s}^1 \left[1 - F(a,1)\right] da$$

Recall also that for $a \in (\underline{a}_s, \overline{a}_s)$, we have:

$$U_A'(a) = U_B'(z(a)) \cdot z'(a), \quad U_A'(a) = z'(a) = s, \quad U_B'(z(a)) = 1.$$

Thus:

$$W[z_s] = \int_{\underline{a}_s}^{\overline{a}_s} U_B'(z(a)) \cdot z'(a) \cdot \left[1 - F\left(a, z(a)\right)\right] da + \int_{\overline{a}_s}^{1} \left[1 - F(a,1)\right] da.$$

Changing variables in the first integral gives:

$$W[z_s] = \int_{\underline{b}_s}^{\overline{b}_s} U_B'(b) \left[1 - F\left(z^{-1}(b), b\right)\right] db + \int_{\overline{a}_s}^{1} \left[1 - F(a,1)\right] da.$$

Now, by Lemma 3 we have the following when $s < 1$:

$$U_A'(a) = \begin{cases} 0, & \text{if } a < \underline{a}_s, \\ s, & \text{if } a \in (\underline{a}_s, \overline{a}_s), \\ 1, & \text{if } a > \overline{a}_s. \end{cases}$$

Moreover, recall that for $a \in (\underline{a}_s, \overline{a}_s)$:

$$U_B'(z(a)) = \frac{U_A'(a)}{z'(a)} = \frac{s}{s} = 1.$$

We therefore have $U_B'(b) = 1$ on $(\underline{b}_s, \overline{b}_s)$. We can then write:

$$\begin{aligned} W[z_s] &= \int_{\underline{b}_s}^{\overline{b}_s} \left[1 - F\left(z^{-1}(b), b\right)\right] db + \int_{\overline{a}_s}^{1} \left[1 - F(a,1)\right] da \\ &= \overline{b}_s - \underline{b}_s - \int_0^1 F\left(\hat{z}^{-1}(b), b\right) db + \int_{\overline{a}_s}^{1} \left[1 - F(a,1)\right] da, \end{aligned}$$

Now, $\overline{b}_s$ increases in $s$ and $\underline{b}_s, \overline{a}_s$ decrease in $s$. Finally, note that an increase in the slope of $z_s$ leads to a decrease in the slope of $z_s^{-1}$. Thus, the second term decreases in $s$ by a result symmetric to Fact 4. Thus, $W[z_s]$ increases in $s$ when $s < 1$.

### C.8 Proof of Lemma 5

By Lemma 4, we can restrict attention to mechanisms with a boundary of slope 1. Lemma 3 tells us that, under the optimal $U_A$, such mechanisms offer only two options: good $A$ with $x = 1$ at an ordeal $c_A$ and good $B$ with $x = 1$ at an ordeal $c_B$. Now, suppose one of the supply constraints (S) is slack for such a mechanism; assume without loss this is the case for good $A$.

Since the mechanism allocates a strictly positive amount of good $A$ but its supply constraint is slack, the ordeal for good $A$ has to be interior: $c_A \in (0,1)$. Now, consider an alternative mechanism with ordeals $c_A - \epsilon$ and $c_B$, respectively. This mechanism improves the utilities of all agents, and strictly so for the positive mass of agents who chose good $A$ under the original mechanism. It therefore suffices to show the alternative mechanism is feasible for $\epsilon > 0$ sufficiently small. Note that the mass of agents who take $A$ under the new mechanism is:

$$\int \mathbb{1}\left[a - (c_A - \epsilon) > \max[0,\ b - c_B]\right]\ dF(a,b),$$

since the set of indifferent agents is zero-mass. Moreover:

$$\lim_{\epsilon\to 0}\int \mathbb{1}\left[a - (c_A - \epsilon) > \max[0,\ b - c_B]\right]\ dF(a,b) = \int \mathbb{1}\left[a - c_A > \max[0,\ b - c_B]\right]\ dF(a,b),$$

which is the mass of agents who got good $A$ under the original mechanism. Since the supply constraint (S) for good $A$ was slack, it remains slack for the alternative one when $\epsilon$ is sufficiently small. Similarly, reducing the ordeal for good $A$ can only relax the supply constraint for good $B$, and thus (S) is satisfied for $\epsilon$ small enough.

### C.9 Proof of Theorem 2

Consider the density defined in (11) and supplies given by $s_A = 1-\zeta-\epsilon$, $s_B = \zeta+\epsilon$. I show that for $\epsilon > 0$ sufficiently small, a mechanism using only ordeals is not optimal. Let us consider mechanisms which do not use damages. Then, by an argument analogous to the proof of Lemma 5, we can restrict attention to mechanisms allocating all the available supply. The only such mechanism takes the form:

$$y(a,b) = B,\ c(a,b) = 1/2 \quad \text{when } b-a > 1/2, \qquad y(a,b) = A,\ c(a,b) = 0 \quad \text{when } b-a < 1/2.$$

Thus, the total welfare from this mechanism is:

$$\int_{\{b-a>1/2\}} \left(b - \tfrac{1}{2}\right) f(a,b)\, d(a,b) \ + \ \int_{\{b-a<1/2\}} a\, f(a,b)\, d(a,b) = \frac{14\epsilon^2 - 9\epsilon + 23}{42} - \frac{\zeta\left(28\epsilon^2 - 46\epsilon + 25\right)}{84(1-\epsilon)},$$

which converges to $\frac{46-25\zeta}{84}$ as $\epsilon \to 0^+$.

Now, set $\zeta = 1/3$. Let us consider an alternative mechanism which offers two options: good $A$ with $x_A = 1$ and good $B$ with $x_B^\epsilon < 1$, with no ordeals for either. For small enough $\epsilon$ there exists $x_B^\epsilon$ for which both supply constraint hold with equality. We can verify that:

$$x_B^\epsilon = \frac{7}{16} - \frac{287}{1024}\epsilon + O(\epsilon^2),$$

and so $x_B^\epsilon \to \frac{7}{16}$ as $\epsilon \to 0^+$. Calculation confirms that the limit welfare from this mechanism is:

$$\lim_{\epsilon\to 0^+}\left[\int_{\{x_B^\epsilon b>a\}} x_B^\epsilon b\, f(a,b)\, d(a,b) \ + \ \int_{\{x_B^\epsilon b<a\}} a\, f(a,b)\, d(a,b)\right] = \frac{17497}{36288} > \frac{46-25\,\zeta}{84} = \frac{113}{252},$$

and so this mechanism dominates the no-damage mechanism for $\epsilon$ sufficiently small.

### C.10 Proof of Proposition 4

Maximizing (E) is equivalent to choosing $q_A, q_B : [0,1]^2 \to [0,1]$ to maximize:

$$\int q_A(a,b) \cdot a + q_B(a,b) \cdot b \;\; dF(a,b),$$

subject to:

$$\int q_A(a,b)\, dF(a,b) \le s_A, \quad \int q_B(a,b)\, dF(a,b) \le s_B, \tag{47}$$

$$q_A(a,b) + q_B(a,b) \le 1 \;\text{ for every } (a,b) \in [0,1]^2. \tag{48}$$

Since $s_A + s_B \le 1$ and a unit mass of types has positive values for both goods, both supply constraints (47) will hold with equality. The objective and constraints are linear so the solution exists and must also maximize:

$$\int q_A(a,b) \cdot (a - \eta_A) + q_B(a,b) \cdot (b - \eta_B) \;\; dF(a,b), \tag{49}$$

subject to (48) for some multipliers $\eta_A, \eta_B \ge 0$. Note also that $\eta_A, \eta_B < 1$. Otherwise, the maximizer of (49) would not allocate one of the goods at all, and we know that supply constraints must hold with equality. Now, notice that $q_A, q_B$ maximize (49) if and only if they satisfy the following almost everywhere:

$$q_A(a,b) = \begin{cases} 1, & \text{if } a - \eta_A > \max\{0,\, b - \eta_B\}, \\ 0, & \text{otherwise,} \end{cases} \quad , \quad q_B(a,b) = \begin{cases} 1, & \text{if } b - \eta_B > \max\{0,\, a - \eta_A\}, \\ 0, & \text{otherwise.} \end{cases}$$

Such an allocation is implemented by a mechanism with no damages and two posted prices equal to $\eta_A, \eta_B$. Finally, since individual demands satisfy the gross substitutes condition, these prices are unique (Kelso Jr and Crawford, 1982; Gul and Stacchetti, 1999). Since Theorem 1 offered each good with an ordeal and allocated the whole supply of both goods, the two ordeals must therefore have been equal to the unique market-clearing prices.

### C.11 Proof of Corollary 2

The argument is analogous to the proof of Theorem 1. First, note that the steps related to implementation, i.e. Proposition 1 and Lemma 2, transfer without any modification as they do not rely on the planner's objective. The proof of Lemma 3 also carries through, as it shows that the optimal implementation guarantees the highest utility profile *pointwise*, and thus the argument is unaffected by the presence of welfare weights. Lemma 5 remains valid for the same reason.

Lemma 1, however, requires modification. As before, Proposition 1 lets us rewrite total welfare (13) in terms of $A, B$-indirect utilities $U_A, U_B$ and their associated boundary $z$ as follows:

$$\int_{\underline{a}}^{1} \int_{0}^{\hat{z}(a)} \lambda(a,v) \cdot g(a,v) dv \cdot U_A(a)\, da + \int_{\underline{b}}^{1} \int_{0}^{\hat{z}^{-1}(b)} \lambda(a,v) \cdot g(v,b) dv \cdot U_B(b)\, db. \tag{50}$$

A sequence of steps analogous to those in the proof of Lemma 1 then yields the following expression for the objective:

$$U_A(1) - \int_0^1 U_A'(a) \cdot \tilde{G}(a, \hat{z}(a)) da, \tag{51}$$

where:

$$\tilde{G}(a,b) := \frac{\int_{[0,a]\times[0,b]} \lambda(v,w) \cdot g(v,w) \ \ d(v,w)}{\int_{[0,1]^2} \lambda(v,w) \cdot g(v,w) \ \ d(v,w)}.$$

A similar modification is required in the proof of Proposition 2. Note that while the objective is then written in terms of $\tilde{G}$, the area constraint is still phrased in terms of the unmodified density $g$. Following the steps of the derivation then yields the analog of (38):

$$\phi''(a) = -\left(\int_0^a \lambda(v, z^*(a)) \cdot g\left(v, z^*(a)\right) dv - \mu \cdot g\left(a, z^*(a)\right)\right). \tag{52}$$

The rest of the original argument carries through exactly, only with the expression

$$\frac{\int_0^a f\left(v, z^*(a)\right) dv}{f\left(a, z^*(a)\right)} = \frac{F_{A|B}\left(a|z^*(a)\right)}{f_{A|B}\left(a|z^*(a)\right)},$$

now replaced with

$$\frac{\int_0^a \lambda(v, z^*(a)) \cdot g\left(v, z^*(a)\right) dv}{g\left(a, z^*(a)\right)}.$$

A similar modification must be made to Lemma 4; the following expression in the original proof

$$\int_{\overline{\mathcal{D}}} \left(\int_0^a g(v,b)\, dv\right) \mathrm{d}(a,b) \ - \ \int_{\underline{\mathcal{D}}} \left(\int_0^a g(v,b)\, dv\right) \mathrm{d}(a,b),$$

is now replaced with

$$\int_{\overline{\mathcal{D}}} \left(\int_0^a \lambda(v,b) \cdot g(v,b)\, dv\right) \mathrm{d}(a,b), \ - \ \int_{\underline{\mathcal{D}}} \left(\int_0^a \lambda(v,b) \cdot g(v,b)\, dv\right) \mathrm{d}(a,b),$$

which can in turn be written as:

$$\int_{\overline{\mathcal{D}}} \frac{\int_0^a \lambda(v,b) \cdot g(v,b)\, dv}{g(a,b)} \cdot g(a,b)\, \mathrm{d}(a,b) \ - \ \int_{\underline{\mathcal{D}}} \frac{\int_0^a \lambda(v,b) \cdot g(v,b)\, dv}{g(a,b)} \cdot g(a,b)\, \mathrm{d}(a,b).$$

The rest of the argument mirrors the original one.

### C.12 Proof of Proposition 5

By an argument analogous to that for Lemma 1, we can express the objective as follows:

$$\int_0^1 \left(1 - F(a, \hat{z}(a)) - r \cdot \tilde{f}(a, \hat{z}(a))\right) \cdot U_A'(a)\, da,$$

where:

$$\tilde{f}(a,b) = \int_0^b f(a,t)dt + \int_0^a f(t,b)dt \cdot \mathbb{1}[b < 1].$$

Then the optimal mechanism has to solve the following problem:

**Problem 5.** *Choose a feasible pair $(z, U_A)$ to maximize:*

$$\int_0^1 \left(1 - F(a, \hat{z}(a)) - r \cdot \tilde{f}(a, \hat{z}(a))\right) \cdot U_A'(a)\, da. \tag{53}$$

To obtain a necessary condition for the solution, we invoke Proposition 7 used previously to prove Lemma 3. Indeed, note that any solution to Problem 5 must also solve Problem 1, and thus, by Lemma 7, we can assume that the optimal pair $(z^*, U_A^*)$ satisfies:

$$U_A^{*\prime}(a) := \begin{cases} 0, & a < \underline{a}, \\ m(a) \cdot k, & \underline{a} < a < \overline{a}, \\ m(\overline{a}) \cdot k, & \overline{a} < a < h, \\ 1, & h < a, \end{cases}, \quad \text{for some } h \geq \overline{a}. \tag{54}$$

Equation (54) thus pins down the structure of $U_A$ optimally implementing any feasible boundary $z$. Indeed, note that, just like in the case without savings from quality reduction, $U_A'(a)$ is constant on intervals where $z$ is concave and proportional to $z'$ on intervals where $z$ is convex.

Let us now consider the shape of the optimal boundary. Following the logic of the proof of Theorem 1, I show that under Assumption 3 the optimal boundary is piecewise linear. Analogously to the original argument, consider any closed interval where $z^*$ is concave and twice continuously differentiable. There, it has to solve a version of Problem 3 where the objective takes the form:

$$-\int_{\underline{v}}^{\overline{v}} F\left(a, z(a)\right) \;-\; r \cdot \left(\int_0^{z(a)} f(a,v)dv + \int_0^a f(v, z(a))dv\right) da. \tag{55}$$

We now apply the Maximum Principle to this problem in a manner analogous to the first part of the proof of Proposition 2. Together with Assumption 3, this gives the following necessary condition for optimality:

$$\mu - r = \frac{F_A(a)}{f_A(a)} + r \cdot \frac{F_A(a)}{f_A(a)} \cdot \frac{f_B'(z(a))}{f_B(z(a))} \quad \text{for } a \in (\underline{v}, \overline{v}).$$

However, this condition cannot hold, as $\frac{F_A}{f_A}$ and $z$ are strictly increasing, and $\frac{f_B'}{f_B}$ is non-decreasing.

The argument for strictly convex regions is symmetric; thus, the optimal boundary $z^*$ must be piecewise linear. We now also show the optimal boundary cannot have kinks. Pick $\overline{v}$ such that $[\underline{a}, \underline{v}]$ is the largest interval starting with $\underline{a}$ on which $z^*$ is convex or concave. Assume w.l.o.g. that $z^*$ is concave on it. There, analogously to the latter part of the proof of Proposition 2, $z^*$ has to solve a version of Problem 4 with the objective (55).

Since $z^*$ is piecewise linear, the largest initial concave interval either covers all of $[\underline{a}, \overline{a}]$, or consists of at least two linear pieces. In what follows, I show that the solution to this problem cannot have jumps, and thus that the latter cannot happen. This in turn proves that $z^*$ is linear.

Suppose such a jump existed at some $a^*$. We can establish, analogously to the proof of Proposition 2, that $\phi$, which is the costate on $z'$, has to satisfy $\phi'(a^*) = 0$ and $\phi''(a^*) \geq 0$ there. However, we can show that:

$$\phi''(a) \; = = f\left(a, z(a)\right)\left(\mu - r - \frac{F_A(a)}{f_A(a)} - r \cdot \frac{F_A(a)}{f_A(a)} \cdot \frac{f_B'(z(a))}{f_B(z(a))}\right).$$

Note that, by Assumption 3, $\phi''(a)$ is either strictly negative everywhere on $(\underline{a}, \overline{v})$ or positive until some $\tilde{a} \in (\underline{a}, \overline{v})$ and then negative forever after. An argument analogous to that in the proof of Proposition 2 then shows we cannot have $\phi'(a^*) = 0$ and $\phi''(a^*) \geq 0$ at any interior $a^*$.

Consequently, the optimal pair $(z^*, U_A^*)$ features a linear boundary. Assume without loss that $z(\overline{a}) = 1$. Let us then consider two cases. If the slope of the optimal boundary is greater than 1, Condition (54) implies that $U_A^{*\prime}$ takes value 1 everywhere. Since we have $U_A'(a) = U_B'(z(a))\,z'(a)$,, we also know $U_B^{*\prime}$ takes the value $1/s$ everywhere. If, on the other hand, the slope of the optimal boundary is below 1, $U_A'$ equals $s$ below some value and equals 1 above it. As shown in the proof of Theorem 1, this implies the optimal mechanism can be implemented with a menu of at most three options.